21

COVARIANCE :

If X and Y are random variables, then covariance between X and Y is defined as,

$$\boxed{Cov(X,Y) = E(XY) - E(X)E(Y)} \rightarrow (1)$$

If X and Y are independent, then

$$E(XY) = E(X) \cdot E(Y) \rightarrow (2)$$

Subs (2) in (1), we get

$$Cov(X,Y) = E(X)E(Y) - E(X)E(Y)$$

$$Cov(X,Y) = 0.$$

$\therefore$ If X and Y are independent, then

$$\boxed{Cov(X,Y) = 0}$$

Note :

1. $Cov(aX, bY) = ab\, Cov(X,Y)$
2. $Cov(X+a, Y+b) = Cov(X,Y)$
3. $Cov(aX+b, cY+d) = ac\, Cov(X,Y)$
4. $Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2\, Cov(X_1, X_2)$
5. $Var(X_1 - X_2) = Var(X_1) + Var(X_2) - 2\, Cov(X_1, X_2)$
6. If X_1 and X_2 are independent,

   $Var(X_1 \pm X_2) = Var(X_1) + Var(X_2)$

(24)

Solution :

(23)

PROBLEMS :

(1) Find the coefficient of correlation between industrial production and export using the following data:

Production(x)	55	56	58	59	60	60	62
Export(Y)	35	38	37	39	44	43	44

Solution:

X	Y	XY	X^2	Y^2
55	35	1925	3025	1225
56	38	2128	3136	1444
58	37	2146	3364	1369
59	39	2301	3481	1521
60	44	2640	3600	1936
60	43	2580	3600	1849
62	44	2728	3844	1936
410	280	16448	24050	11280

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{\Sigma XY}{n} - \frac{\Sigma X}{n} \frac{\Sigma Y}{n}$$

$$= \frac{16448}{7} - \frac{410}{7} \cdot \frac{280}{7}$$

$$\mathrm{Cov}(X,Y) = 2349.714 - 2342.857$$

$$\boxed{\mathrm{Cov}(X,Y) = 6.8569}$$

$$\sigma_X = \sqrt{\frac{\Sigma X^2}{n} - \left(\frac{\Sigma X}{n}\right)^2}$$

$$= \sqrt{\frac{24050}{7} - \left(\frac{410}{7}\right)^2}$$

$$\boxed{\sigma_X = 2.2588}$$

$$\sigma_Y = \sqrt{\frac{\Sigma Y^2}{n} - \left(\frac{\Sigma Y}{n}\right)^2}$$

$$= \sqrt{\frac{11280}{7} - \left(\frac{280}{7}\right)^2}$$

$$\boxed{\sigma_Y = 3.3806}$$

$$r(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \cdot \sigma_Y}$$

$$= \frac{6.8569}{2.2588 \times 3.3806}$$

$$\boxed{r(X,Y) = 0.898}$$

---

(2) Two random variables X and Y have the following joint Probability density function

$$f(x,y) = \begin{cases} 2-x-y, & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find $\mathrm{Var}(X)$, $\mathrm{Var}(Y)$ and also the Covariance between X and Y. Also find ρ_{XY}.

(24)

Solution :

First, let us find the marginal pdf of X and Y.

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_0^1 (2-x-y)\,dy$$

$$= \left[2y - xy - \frac{y^2}{2}\right]_0^1$$

$$= 2 - x - \frac{1}{2}$$

$$\boxed{f(x) = \frac{3}{2} - x} \quad \text{where } 0 \le x \le 1$$

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^1 (2-x-y)\,dx$$

$$= \left[2x - \frac{x^2}{2} - xy\right]_0^1$$

$$= 2 - \frac{1}{2} - y$$

$$\boxed{f(y) = \frac{3}{2} - y} \quad \text{where } 0 \le y \le 1$$

$$E(X) = \int_{-\infty}^{\infty} x\,f(x)\,dx$$

$$= \int_0^1 x\left(\frac{3}{2} - x\right) dx$$

$$= \int_0^1 \left(\frac{3}{2}x - x^2\right) dx$$

(or)

$$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,f(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^1 x(2-x-y)\,dx\,dy$$

$$= \int_0^1\int_0^1 (2x - x^2 - xy)\,dx\,dy$$

$$= \int_0^1 \left[\frac{2x^2}{2} - \frac{x^3}{3} - \frac{x^2 y}{2}\right]_0^1 dy$$

$$= \int_0^1 \left[1 - \frac{1}{3} - \frac{y}{2}\right] dy$$

$$= \left[y - \frac{y}{3} - \frac{y^2}{4}\right]_0^1$$

$$= 1 - \frac{1}{3} - \frac{1}{4} = \frac{12-4-3}{12}$$

$$= 5/12$$

$$E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\,f(x,y)\,dx\,dy$$

$$E(Y) = \frac{5}{12}$$

$$E(X^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f(x,y)\,dx\,dy$$

$$= \frac{1}{4}$$

$$E(Y^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f(x,y)\,dx\,dy$$

$$= \frac{1}{4}$$

(26)

$$E(x) = 4a\left[\frac{x^3}{3}\right]_0^1$$

(25)

$$Var(Y) = E(Y^2) - [E(Y)]^2$$

$$= \frac{1}{4} - \left(\frac{5}{12}\right)^2$$

$$Var(Y) = \frac{11}{144}$$

'F

$$\sigma_X^2 = \frac{11}{144} \Rightarrow \sigma_X = \frac{\sqrt{11}}{12}$$

$$\sigma_Y^2 = \frac{11}{144} \Rightarrow \sigma_Y = \frac{\sqrt{11}}{12}$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^1 xy(2-x-y)\,dx\,dy$$

$$= \int_0^1\int_0^1 (2xy - x^2y - xy^2)\,dx\,dy$$

$$= \int_0^1 \left[\frac{2x^2y}{2} - \frac{x^3y}{3} - \frac{x^2y^2}{2}\right]_0^1 dy$$

$$= \int_0^1 \left[y - \frac{y}{3} - \frac{y^2}{2}\right]_0^1 dy$$

$$= \left[\frac{y^2}{2} - \frac{y^2}{6} - \frac{y^3}{6}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{6} - \frac{1}{6}$$

$$\boxed{E(XY) = \frac{1}{6}}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{1}{6} - \frac{5}{12} \cdot \frac{5}{12}$$

$$= \frac{1}{6} - \frac{25}{144}$$

$$\boxed{Cov(X,Y) = \frac{-1}{144}}$$

$$r(X,Y) = \rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

$$= \frac{-1/144}{\frac{\sqrt{11}}{12} \cdot \frac{\sqrt{11}}{12}} = \frac{-1}{11}$$

$$\boxed{r(X,Y) = \frac{-1}{11}}$$

---

3) The independent variables X and Y have the probability density functions given by,

$$f_X(x) = \begin{cases} 4ax, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 4by, & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the correlation coefficient between X and Y.

Solution :

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{0}^{1} x\,(4ax)\, dx$$