Green's Theorem:

If $M, N, \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ are continuous and one-valued functions in a region R enclosed by the curve C, then

$$\int_C [M\,dx + N\,dy] = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$

problems:

1. Evaluate $\int_C (x^2 + xy)\,dx + (x^2 + y^2)\,dy$, where C is the square bounded by the lines $x=0$, $x=1$, $y=0$ and $y=1$.

Soln.

Green's theorem:

$$\int_C [M\,dx + N\,dy] = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$

Here $M = xy + x^2$ | $N = x^2 + y^2$

$\frac{\partial M}{\partial y} = x$ | $\frac{\partial N}{\partial x} = 2x$

Now,

$$\int_C [M\,dx + N\,dy] = \iint_R [2x - x]\,dx\,dy$$

$$= \int_0^1 \int_0^1 x\,dx\,dy = \int_0^1 \left[\frac{x^2}{2}\right]_0^1 dy$$

$$= \int_0^1 \left[\frac{1}{2} - 0\right] dy$$

$$= \frac{1}{2}[y]_0^1$$

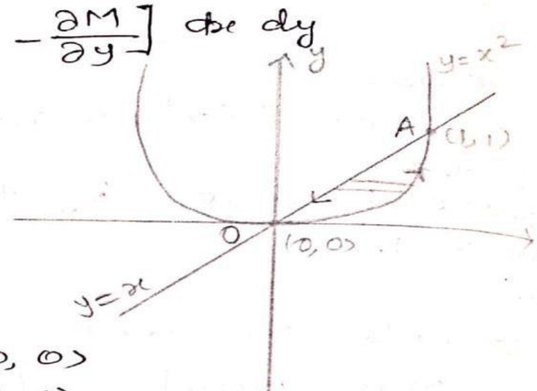
$$\int_C [(x^2 + xy)\,dx + (x^2 + y^2)\,dy] = \frac{1}{2}$$

2) Verify Green's theorem for $\int_C (xy+y^2)\,dx + x^2\,dy$ where C is the closed curve bounded by $y=x^2$ and $y=x$.

Soln.

By Green's theorem,

$$\int_C [M\,dx + N\,dy] = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$



Given $y = x^2$; $y = x$

$\Rightarrow x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x=0,\ x=1$

when $x=0,\ y=0 \Rightarrow (0,0)$

$x=1,\ y=1 \Rightarrow (1,1)$

Here $M = xy + y^2$ | $N = x^2$

$\frac{\partial M}{\partial y} = x + 2y$ | $\frac{\partial N}{\partial x} = 2x$

RHS

$$\iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy = \int_0^1 \int_y^{\sqrt{y}} [2x - (x+2y)]\,dx\,dy$$

$$= \int_0^1 \int_y^{\sqrt{y}} [x - 2y]\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^2}{2} - 2xy\right]_{x=y}^{\sqrt{y}} dy$$

$$= \int_0^1 \left[\left(\frac{y}{2} - 2y^{3/2}\right) - \left(\frac{y^2}{2} - 2y^2\right)\right] dy$$

$$= \int_0^1 \left[\frac{y}{2} - 2y^{3/2} + \frac{3}{2}y^2\right] dy$$

$$= \left[\frac{y^2}{4} - 2\frac{y^{5/2}}{5/2} + \frac{3}{2}\frac{y^3}{3} \right]_0^1$$

$$= \left(\frac{1}{4} - \frac{4}{5} + \frac{1}{2}\right) - 0$$

$$= \frac{5 - 16 + 10}{20}$$

$$\text{RHS} = \frac{-1}{20}$$

LHS

To evaluate $\int_C [M\,dx + N\,dy]$, we shall take C in the different paths.

i). along AO $[y = x]$

ii). along OA $[y = x^2]$

Along with AO $[y = x \Rightarrow dy = dx]$

$$\int_{AO} [M\,dx + N\,dy] = \int_{AO} [(xy + y^2)\,dx + x^2\,dy]$$

$$= \int_1^0 [(x^2 + x^2)\,dx + x^2\,dx]$$

$$= \int_1^0 [x^2 + x^2 + x^2]\,dx$$

$$= 3\int_1^0 x^2\,dx$$

$$= 3\left[\frac{x^3}{3}\right]_1^0$$

$$= [0 - 1]$$

$$= -1$$

Along with OA $[y = x^2 \Rightarrow dy = 2x\,dx]$

$$\int_{OA} [(xy + y^2)\,dx + x^2\,dy] = \int_0^1 [(x(x^2) + x^4)\,dx + x^2(2x\,dx)]$$

$$= \int_0^1 [x^3 + x^4 + 2x^3]\, dx$$

$$= \left[\frac{x^4}{4} + \frac{x^5}{5} + \frac{2x^4}{4}\right]_0^1$$

$$= \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{2}\right) - 0$$

$$= \frac{5+4+10}{20}$$

$$= \frac{19}{20}$$

$$\therefore \int_C (M\,dx + N\,dy) = \int_{OA} + \int_{AO} = \frac{19}{20} - 1$$

$$= \frac{19-20}{20}$$

$$\int_C (M\,dx + N\,dy) = \frac{-1}{20}$$

$\therefore$ LHS = RHS

Hence green's theorem is verified.

3]. Verify green's theorem for $\int_C (x^2 - y^2)\,dx + 2xy\,dy$ where C is the closed curve bounded by $y = x^2$ and $y^2 = x$

Soln.

Given $y = x^2$ and $y^2 = x$

$$\Rightarrow y = (y^2)^2$$

$$y^4 - y = 0$$

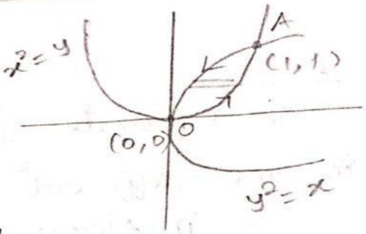
$$y(y^3 - 1) = 0$$

$$y = 0, \quad y^3 - 1 = 0$$

$$y^3 = 1 \Rightarrow y = 1$$

when $y=0 \Rightarrow x=0 \Rightarrow (0,0)$

$y=1 \Rightarrow x=1 \Rightarrow (1,1)$



By Green's Theorem,

$$\int_C M\,dx+N\,dy = \iint_R \left[\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right] dx\,dy$$

Here $M = x^2-y^2$ | $N = 2xy$

$\frac{\partial M}{\partial y} = -2y$ | $\frac{\partial N}{\partial x} = 2y$

RHS

$$\iint_R \left[\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right] dx\,dy = \int_0^1\int_{y^2}^{\sqrt{y}} \left[2y+2y\right] dx\,dy$$

$$= \int_0^1 4y(x)_{y^2}^{\sqrt{y}}\,dy$$

$$= \int_0^1 \left[4yy^{1/2} - 4yy^2\right] dy = \int_0^1 \left[4y^{\frac{3}{2}} - 4y^3\right] dy$$

$$= 4\left[\frac{y^{3/2+1}}{\frac{3}{2}+1} - \frac{y^4}{4}\right]_0^1$$

$$= 4\left[\frac{2}{5}y^{5/2} - \frac{y^4}{4}\right]_0^1 = 4\left[\frac{2}{5}-\frac{1}{4}\right]$$

$$= 4\left[\frac{8-5}{20}\right]$$

$$= \frac{12}{20}$$

$$\iint_R \left[\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right] dx\,dy = \frac{3}{5}$$

LHS

To evaluate $\int_C [M\,dx + N\,dy]$, we shall take c in the different paths.

i). Along OA $[y = x^2]$

ii). Along AO $[y^2 = x]$

Along OA $[y = x^2 \Rightarrow dy = 2x\,dx]$

$$\int_{OA} [(x^2 - y^2)\,dx + 2xy\,dy]$$

$$= \int_0^1 [(x^2 - x^4)\,dx + 2x(x^2)(2x\,dx)]$$

$$= \int_0^1 [x^2 - x^4 + 4x^4]\,dx$$

$$= \int_0^1 [3x^4 + x^2]\,dx = \left[\frac{3x^5}{5} + \frac{x^3}{3}\right]_0^1$$

$$= \left(\frac{3}{5} + \frac{1}{3}\right) - 0$$

$$= \frac{9+5}{15} - 0$$

$$= \frac{14}{15}$$

Along AO $[y^2 = x \Rightarrow 2y\,dy = dx \Rightarrow dy = \frac{dx}{2\sqrt{x}}$

$$\int_{AO} [(x^2 - y^2)\,dx + 2xy\,dy]$$

$$= \int_1^0 \left[(x^2 - x)\,dx + 2x\,x^{1/2}\frac{dx}{2\sqrt{x}}\right]$$

$$= \int_1^0 [(x^2 - x) + x]\,dx = \int_1^0 x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_1^0 = 0 - \frac{1}{3} = -\frac{1}{3}$$

Now,

$$\int_C (x^2-y^2)\,dx + 2xy\,dy = \int_{OA} + \int_{AO}$$

$$= \frac{14}{15} - \frac{1}{3}$$

$$= \frac{14-5}{15}$$

$$= \frac{9}{15}$$

$$= \frac{3}{5}$$

$\therefore$ LHS = RHS

Hence verified.