Unit - II

Vector calculus

Gradient:

Let $\phi(x, y, z)$ be a scalar point function and is continuously differentiable. Then the vector

$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$ is called the gradient of the scalar fn. ϕ.

i.e., $\text{grad}\,\phi = \nabla\phi$

Problems

1] Find $\nabla\phi$ where $\phi = x^2+y^2+z^2$

Soln.

$$\text{Grad}\,\phi \text{ (or) } \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2+y^2+z^2) + \vec{j}\frac{\partial}{\partial y}(x^2+y^2+z^2) + \vec{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$= \vec{i}(2x) + \vec{j}(2y) + \vec{k}(2z)$$

$$\nabla\phi = 2x\vec{i} + 2y\vec{j} + 2z\,k$$

2]. Find $\nabla\phi$ where $\phi = 3x^2y - y^3z^2$ at $(1, 1, 1)$

Soln.

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \vec{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \vec{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= \vec{i}[6xy - 0] + \vec{j}[3x^2 - 3y^2z^2] + \vec{k}[0 - 2y^3z]$$

$$\nabla\phi = 6xy\,\vec{i} + (3x^2 - 3y^2z^2)\vec{j} - 2y^3z\,\vec{k}$$

$$\nabla\phi_{(1,1,1)} = 6(1)(1)\vec{i} + (3-3)\vec{j} - 2(1)(1)\vec{k}$$

$$= 6\vec{i} + 0\vec{j} - 2\vec{k}$$

$$= 6\vec{i} - 2\vec{k}$$

3]. Find the maximum directional derivative of $\phi = xyz^2$ at $(1, 0, 3)$

Soln.

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(xyz^2) + \vec{j}\frac{\partial}{\partial y}(xyz^2) + \vec{k}\frac{\partial}{\partial z}(xyz^2)$$

$$\nabla\phi = \vec{i}(yz^2) + \vec{j}(xz^2) + \vec{k}(xy\,2z)$$

$$\nabla\phi_{(1,0,3)} = \vec{i}(0) + \vec{j}(1)(9) + \vec{k}(0)$$

$$= 9\vec{j} \qquad \text{maximum DD} = \sqrt{81} = 9$$

4]. Find $\nabla\phi$ where $\phi = xyz$ at $(1, 2, 3)$

Soln.

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial(xyz)}{\partial x} + \vec{j}\frac{\partial(xyz)}{\partial y} + \vec{k}\frac{\partial(xyz)}{\partial z}$$

$$\nabla\phi = \vec{i}(yz) + \vec{j}(xz) + \vec{k}(xy)$$

$$\nabla\phi)_{(1,2,3)} = \vec{i}(2)(3) + \vec{j}(1)(3) + \vec{k}(1)(2)$$

$$= 6\vec{i} + 3\vec{j} + 2\vec{k}$$

5]. If $\nabla\phi = yz\vec{i} + zx\vec{j} + xy\vec{k}$, find ϕ.

Soln.

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = \vec{i}(yz) + \vec{j}(zx) + \vec{k}(xy)$$

Equating w.r. to $\vec{i}, \vec{j}, \vec{k}$

$\frac{\partial\phi}{\partial x} = yz$	$\frac{\partial\phi}{\partial y} = zx$	$\frac{\partial\phi}{\partial z} = xy$
Integrate w.r. to x	w.r. to y	w.r. to z
$\phi = xyz + f(y, z)$	$\phi = xyz + f(x, z)$	$\phi = xyz + f(x, y)$

In general, $\phi = xyz + C$

6). If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, such that $|\vec{r}| = r$, prove that

i). $\nabla r = \frac{\vec{r}}{r} = \hat{r}$

ii) $\nabla\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3} = \frac{-\hat{r}}{r^2}$

iii). $\nabla r^n = n r^{n-2}\vec{r}$

iv). $\nabla f(r) = f'(r)\nabla r$

Soln.

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\Rightarrow r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

$\Rightarrow r^2 = x^2+y^2+z^2 \rightarrow (1)$

Diff. (1) w.r. to x, y, z,

$2r\frac{\partial r}{\partial x} = 2x$	$2r\frac{\partial r}{\partial y} = 2y$	$2r\frac{\partial r}{\partial z} = 2z$
$\frac{\partial r}{\partial x} = \frac{x}{r}$	$\frac{\partial r}{\partial y} = \frac{y}{r}$	$\frac{\partial r}{\partial z} = \frac{z}{r}$

i). $\nabla r = \vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}$

$= \vec{i}\left(\frac{x}{r}\right) + \vec{j}\left(\frac{y}{r}\right) + \vec{k}\left(\frac{z}{r}\right)$

$= \frac{x\vec{i} + y\vec{j} + z\vec{k}}{r}$

$\nabla r = \frac{\vec{r}}{r}$

ii) $\nabla\left(\frac{1}{r}\right) = \vec{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \vec{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \vec{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$

$= \vec{i}\left[-\frac{1}{r^2}\frac{\partial r}{\partial x}\right] + \vec{j}\left[-\frac{1}{r^2}\frac{\partial r}{\partial y}\right] + \vec{k}\left[\frac{-1}{r^2}\frac{\partial r}{\partial z}\right]$

$= \vec{i}\left[\frac{-1}{r^2}\times\frac{x}{r}\right] + \vec{j}\left[-\frac{1}{r^2}\times\frac{y}{r}\right] + \vec{k}\left[\frac{-1}{r^2}\times\frac{z}{r}\right]$

$= -\frac{1}{r^3}\left[x\vec{i} + y\vec{j} + z\vec{k}\right]$

$\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$

iii). $\nabla r^n = \vec{i}\dfrac{\partial (r^n)}{\partial x} + \vec{j}\dfrac{\partial (r^n)}{\partial y} + \vec{k}\dfrac{\partial (r^n)}{\partial z}$

$= \vec{i}\, n r^{n-1}\dfrac{\partial r}{\partial x} + \vec{j}\, n r^{n-1}\dfrac{\partial r}{\partial y} + \vec{k}\, n r^{n-1}\dfrac{\partial r}{\partial z}$

$= n r^{n-1}\left[\vec{i}\dfrac{\partial r}{\partial x} + \vec{j}\dfrac{\partial r}{\partial y} + \vec{k}\dfrac{\partial r}{\partial z}\right]$

$= n r^{n-1}\left[\vec{i}\left(\dfrac{x}{r}\right) + \vec{j}\left(\dfrac{y}{r}\right) + \vec{k}\left(\dfrac{z}{r}\right)\right]$

$= \dfrac{n r^{n-1}}{r}\left[x\vec{i} + y\vec{j} + z\vec{k}\right]$

$= \dfrac{n r^{n-1}}{r}\vec{r}$

$\nabla r^n = n r^{n-2}\vec{r}$

iv). $\nabla f(r) = \vec{i}\dfrac{\partial}{\partial x}f(r) + \vec{j}\dfrac{\partial}{\partial y}f(r) + \vec{k}\dfrac{\partial}{\partial z}f(r)$

$= \vec{i}f'(r)\dfrac{\partial r}{\partial x} + \vec{j}f'(r)\dfrac{\partial r}{\partial y} + \vec{k}f'(r)\dfrac{\partial r}{\partial z}$

$= f'(r)\left[\vec{i}\left(\dfrac{x}{r}\right) + \vec{j}\left(\dfrac{y}{r}\right) + \vec{k}\left(\dfrac{z}{r}\right)\right]$

$= \dfrac{f'(r)}{r}\left[x\vec{i} + y\vec{j} + z\vec{k}\right]$

$= f'(r) \times \dfrac{\vec{r}}{r}$

$\nabla f(r) = f'(r)\,\nabla r$ (from (i))

Surfaces :

Unit normal vector $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

Normal derivative $= |\nabla\phi|$

Directional derivative $= \nabla\phi \cdot \dfrac{\vec{a}}{|\vec{a}|}$

Angle between the surfaces :

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}$$

If any two surfaces are cut orthogonally, then $\nabla\phi_1 \cdot \nabla\phi_2 = 0$

1. Find the unit normal to the surface $x^2+xy+z^2=4$ at $(1,-1,2)$.

Soln.

Let $\phi = x^2+xy+z^2-4$

unit normal vector $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

Now
$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$
$$= \vec{i}\frac{\partial}{\partial \phi}(x^2+xy+z^2-4) + \vec{j}\frac{\partial}{\partial y}(x^2+xy+z^2-4) + \vec{k}\frac{\partial}{\partial z}(x^2+xy+z^2-4)$$
$$= \vec{i}(2x+y) + \vec{j}(x) + \vec{k}(2z)$$
$$\nabla\phi_{(1,-1,2)} = \vec{i}(2(1)-1) + \vec{j}(1) + \vec{k}(2(2))$$
$$= \vec{i}+\vec{j}+4\vec{k}$$

$$\therefore \hat{n} = \frac{\vec{i}+\vec{j}+4\vec{k}}{\sqrt{1+1+16}} = \frac{\vec{i}+\vec{j}+4\vec{k}}{\sqrt{18}}$$

2. Find the directional derivative of $\phi = xyz$ at $(1,1,1)$ in the direction of $\vec{i}+\vec{j}+\vec{k}$

Soln.

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = \vec{i}(yz) + \vec{j}(xz) + \vec{k}(xy)$$

$$\nabla\phi_{(1,1,1)} = \vec{i}(1)(1) + \vec{j}(1)(1) + \vec{k}(1)(1)$$

$$= \vec{i} + \vec{j} + \vec{k}$$

Given $\vec{a} = \vec{i} + \vec{j} + \vec{k}$

$$|\vec{a}| = \sqrt{1+1+1} = \sqrt{3}$$

$$DD = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= (\vec{i} + \vec{j} + \vec{k}) \cdot \frac{(\vec{i} + \vec{j} + \vec{k})}{\sqrt{3}}$$

$$= \frac{1+1+1}{\sqrt{3}} = \frac{3}{\sqrt{3}}$$

$$DD = \sqrt{3}$$

3. Find the directional derivative of $\phi = x^2 + 2xy$ at $(1,-1,3)$ in the direction of $\vec{i} + 2\vec{j} + 2\vec{k}$

Soln.

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}(2x+2y) + \vec{j}(2x) + \vec{k}(0)$$

$$\nabla\phi = (2x+2y)\vec{i} + 2x\vec{j}$$

$$\nabla\phi_{(1,-1,3)} = (2(1)+2(-1))\vec{i} + 2(-1)\vec{j}$$

$$= -2\vec{j}$$

$$\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$$

$$|\vec{a}| = \sqrt{1+4+4}$$

$$= \sqrt{9} = 3$$

$$DD = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} = -2\vec{j} \cdot \frac{\vec{i} + 2\vec{j} + 2\vec{k}}{3}$$

$$= -\frac{4}{3}$$

4]. What is the greatest rate of increase of $u = x^2 + yz^2$ at $(1, -1, 3)$

Soln.

$$\nabla u = \vec{i}\frac{\partial u}{\partial x} + \vec{j}\frac{\partial u}{\partial y} + \vec{k}\frac{\partial u}{\partial z}$$

$$= \vec{i}(2x) + \vec{j}(z^2) + \vec{k}(2yz)$$

$$\nabla u = 2x\vec{i} + z^2\vec{j} + 2yz\vec{k}$$

$$\nabla u_{(1,-1,3)} = 2\vec{i} + 9\vec{j} + 2(-1)(3)\vec{k}$$

$$= 2\vec{i} + 9\vec{j} - 6\vec{k}$$

$\therefore$ The greatest rate increase in the direction of y.

5]. Find the angle b/w the normals to the surface $xy = z^2$ at the points $(1, 4, 2)$ & $(-3, -3, 3)$

Soln.

Given $xy = z^2$

$$\phi = xy - z^2$$

$$\nabla\phi = \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}$$

$$= \vec{i}(y) + \vec{j}(x) + \vec{k}(-2z)$$

$$= y\vec{i} + x\vec{j} - 2z\vec{k}$$

$$\nabla\phi_{1\,(1,4,2)} = 4\vec{i} + \vec{j} - 4\vec{k}$$

$$|\nabla\phi_1| = \sqrt{16+1+16} = \sqrt{33}$$

and $$\nabla\phi_{2\,(-3,-3,3)} = -3\vec{i} - 3\vec{j} - 6\vec{k}$$

$$|\nabla\phi_2| = \sqrt{9+9+36} = \sqrt{54} = 3\sqrt{6}$$

$$\therefore \cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}$$

$$= \frac{(4\vec{i} + \vec{j} - 4\vec{k}) \cdot (-3\vec{i} - 3\vec{j} - 6\vec{k})}{\sqrt{33}\;3\sqrt{6}}$$

$$= \frac{-12 - 3 + 24}{3\sqrt{11 \times 3 \times 3 \times 2}} = \frac{9}{3 \times 3\sqrt{22}} = \frac{1}{\sqrt{22}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{22}}\right)$$

6). Find the angle between the surfaces $x^2 - y^2 - z^2 = 11$ and $xy + yz - zx = 18$ at $(6, 4, 3)$

Soln.

Let $\phi_1 = x^2 - y^2 - z^2 - 11$

$$\nabla\phi_1 = \vec{i}\frac{\partial\phi_1}{\partial x} + \vec{j}\frac{\partial\phi_1}{\partial y} + \vec{k}\frac{\partial\phi_1}{\partial z}$$

$$= \vec{i}(2x) + \vec{j}(-2y) + \vec{k}(-2z)$$

$$\nabla\phi_{1(6,4,3)} = 12\vec{i} - 8\vec{j} - 6\vec{k} \Rightarrow |\nabla\phi_1| = \sqrt{144+64+36} = \sqrt{244}$$

and $\phi_2 = xy + yz - zx - 18$

$$\nabla\phi_2 = \vec{i}\frac{\partial\phi_2}{\partial x} + \vec{j}\frac{\partial\phi_2}{\partial y} + \vec{k}\frac{\partial\phi_2}{\partial z}$$

$$= \vec{i}(y - z) + \vec{j}(x + z) + \vec{k}(y - x)$$

$$\nabla\phi_{2(6,4,3)} = \vec{i} + 9\vec{j} - 2\vec{k} \Rightarrow |\nabla\phi_2| = \sqrt{1+81+4} = \sqrt{86}$$

$$\therefore \cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$$= \frac{(12\vec{i} - 8\vec{j} - 6\vec{k}) \cdot (\vec{i} + 9\vec{j} - 2\vec{k})}{\sqrt{244}\sqrt{86}}$$

$$= \frac{12(1) - 8(9) - 6(-2)}{\sqrt{244}\sqrt{86}}$$

$$= \frac{-48}{2\sqrt{61}\sqrt{86}}$$

$$\cos\theta = \frac{-24}{\sqrt{5246}}$$

$$\theta = \cos^{-1}\left[\frac{-24}{\sqrt{5246}}\right]$$

7). Find a and b, such that the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at $(1, -1, 2)$

Soln.

Let $\phi_1 = ax^2 - byz - (a+2)x \longrightarrow (1)$

$$\nabla\phi_1 = \vec{i}\frac{\partial\phi_1}{\partial x} + \vec{j}\frac{\partial\phi_1}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi_1 = \vec{i}\,[2ax - (a+2)] + \vec{j}\,[-bz] + \vec{k}\,[-by]$$

$$\nabla\phi_{1\,(1,-1,2)} = \vec{i}\,[2a - a - 2] - 2b\vec{j} + b\vec{k}$$

$$= (a-2)\vec{i} - 2b\vec{j} + b\vec{k}$$

and $\phi_2 = 4x^2y + z^3 - 4$

$$\nabla\phi_2 = 8xy\,\vec{i} + 4x^2\vec{j} + 3z^2\,\vec{k}$$

$$\nabla\phi_{2\,(1,-1,2)} = -8\vec{i} + 4\vec{j} + 12\,\vec{k}$$

Given two surfaces are cut orthogonally.

ie, $\nabla\phi_1 \cdot \nabla\phi_2 = 0$

$$\left[(a-2)\vec{i} - 2b\vec{j} + b\vec{k}\right]\cdot\left[-8\vec{i} + 4\vec{j} + 12\,\vec{k}\right] = 0$$

$$-8(a-2) - 8b + 12b = 0$$

$$-8a + 16 - 8b + 12b = 0$$

$$-2a + b + 4 = 0$$

ie, $2a - b - 4 = 0 \longrightarrow (2)$

Since $(1,-1,2)$ lies on the surface using (1).

$\phi_1(x,y,z) = 0$

$a(1)^2 - b(-1)(2) = (a+2)(1)$

$a + 2b - a - 2 = 0$

$2b = 2 \Rightarrow \boxed{b = 1}$

$(2) \Rightarrow 2a - 1 - 4 = 0 \Rightarrow 2a = 5 \Rightarrow \boxed{a = 5/2}$