Solenoidal vector :-

A vector $\vec{F}$ is said to be solenoidal vector if $\operatorname{div}\vec{F} = 0$

Irrotational vector :-

A vector $\vec{F}$ is said to be irrotational if $\nabla \times \vec{F} = 0$

ie $\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0$

Conservative vector Field:

If a vector point function $\vec{F}$ is expressible as the gradient of a scalar point function φ, then $\vec{F}$ is conservative ie) $\vec{F}$ is conservative if $\vec{F} = \nabla\varphi$. Here φ is called scalar potential. $\vec{F}$ is conservative force if $\operatorname{curl}\vec{F} = 0$.

5. Prove that $\operatorname{div}(\vec{u}\times\vec{v}) = \vec{v}\cdot \operatorname{curl}\vec{u} - \vec{u}\ \operatorname{curl}\vec{v}$

$$\operatorname{div}(\vec{u}\times\vec{v}) = \Sigma\vec{i}\,\frac{\partial}{\partial x}(\vec{u}\times\vec{v})$$
$$= \Sigma\vec{i}\left[\vec{u}\times\frac{\partial\vec{v}}{\partial x} + \frac{\partial\vec{u}}{\partial x}\times\vec{v}\right]$$
$$= \Sigma\vec{i}\left(\frac{\partial\vec{u}}{\partial x}\times\vec{v}\right) + \Sigma\vec{i}\left(\vec{u}\times\frac{\partial\vec{v}}{\partial x}\right)$$
$$= \left(\Sigma\vec{i}\times\frac{\partial\vec{u}}{\partial x}\right)\cdot\vec{v} - \left(\Sigma\vec{i}\times\frac{\partial\vec{v}}{\partial x}\right)\cdot\vec{u}$$
$$= \operatorname{curl}\vec{u}\cdot\vec{v} - \operatorname{curl}\vec{v}\cdot\vec{u}$$
$$= \vec{v}\operatorname{curl}\vec{u} - \vec{u}\cdot\operatorname{curl}\vec{v}.$$

6. Prove that $r^n\vec{r}$ is solenoidal if $n=-3$ & $r^n\vec{r}$ is irrotational for all values of n.

$$r^n\vec{r} = r^n(x\vec{i}+y\vec{j}+z\vec{k})$$
$$= r^n x\vec{i} + r^n y\vec{j} + r^n z\vec{k}$$
$$\operatorname{div}(r^n\vec{r}) = \nabla\cdot(r^n\vec{r})$$
$$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

Now $r^2 = x^2+y^2+z^2$.

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}\ ;\quad 2r\frac{\partial r}{\partial y} = 2y\ ;\quad 2r\frac{\partial r}{\partial z} = 2z$$
$$\frac{\partial r}{\partial y} = \frac{y}{r} \qquad \frac{\partial r}{\partial z} = \frac{z}{r}$$

Now, $\frac{\partial}{\partial x}(r^n x) = x(nr^{n-1})\frac{\partial r}{\partial x} + r^n$

$$= xnr^{n-1}\frac{x}{r} + r^n = x^2nr^{n-2} + r^n$$

lly $\frac{\partial}{\partial y}(r^n y) = y^2nr^{n-2}+r^n$ & $\frac{\partial}{\partial z}(r^n z) = z^2nr^{n-2}+r^n$.

$$\text{div}(r^n\vec{r}) = x^2 n r^{n-2} + r^n + y^2 n r^{n-2} + r^n + z^2 n r^{n-2} + r^n$$

$$= 3r^n + n r^{n-2}(x^2+y^2+z^2)$$

$$= 3r^n + n r^{n-2}(r^2)$$

$$= 3r^n + n r^n = (3+n)r^n$$

The vector $r^n\vec{r}$ is solenoidal if, $\text{div}(r^n\vec{r}) = 0$

$$\Rightarrow (n+3)r^n = 0$$

$$n+3 = 0 \Rightarrow n = -3$$

$\therefore r^n\vec{r}$ is solenoidal if $n = -3$

$$\text{curl}(r^n\vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \Sigma \vec{i}\left[\frac{\partial}{\partial y} r^n z - \frac{\partial}{\partial z} r^n y\right]$$

$$= \Sigma \vec{i}\left\{n r^{n-1}\frac{\partial r}{\partial y} z - n r^{n-1}\frac{\partial r}{\partial z} y\right]$$

$$= \Sigma \vec{i}\left[n r^{n-1}\frac{y}{r} z - n r^{n-1}\frac{z}{r} y\right]$$

$$= \Sigma \vec{i}\, n r^{n-2}[yz - zy]$$

$$= 0.$$

$\therefore \text{curl}(r^n\vec{r}) = 0 \ \forall$ values of n

4. Find the constant a, b, c & the vector $\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ is irrotational.

$\because \vec{F}$ is irrotational $\Rightarrow \nabla\times\vec{F} = 0$.

$$\Rightarrow \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0$$

$\Rightarrow \vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2) = 0$

$\Rightarrow$ Each component is equal to zero

$c+1=0$	$4-a=0$	$b-2=0$
$\Rightarrow c=-1$	$\Rightarrow a=4$	$\Rightarrow b=2$

8) Find 'a' so that the vector $\vec{A} = (ax^2 - y^2 + x)\vec{i} - (2xy + y)\vec{j}$ is irrotational

$\because \vec{A}$ is irrotational $\Rightarrow \nabla \times \vec{A} = 0.$

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax^2 - y^2 + x & -2xy - y & 0 \end{vmatrix} = 0.$$

$\vec{i}[0-0] - \vec{j}[0-0] + \vec{k}[-2y+2y] = 0$

$\because$ there is no 'a' value in the above equation

'a' is arbitrary.

9. Prove $\vec{F} = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$ is irrotational and find its scalar potential $\varphi \ni \vec{F} = \nabla\varphi$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 \end{vmatrix}$$

$= \vec{i}[0-0] - \vec{j}[3z^2 - 3z^2] + \vec{k}[2y\cos x - 2y\cos x]$

$= 0$

$\Rightarrow \vec{F}$ is irrotational

To find φ

$$\nabla\varphi = (y^2\cos x + z^2)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$$

wkt, $\nabla\varphi = \frac{\partial\varphi}{\partial x}\vec{i} + \frac{\partial\varphi}{\partial y}\vec{j} + \frac{\partial\varphi}{\partial z}\vec{k}$

$\Rightarrow \frac{\partial\varphi}{\partial x} = y^2\cos x + z^3$ | $\frac{\partial\varphi}{\partial y} = 2y\sin x - 4$ | $\frac{\partial\varphi}{\partial z} = 3xz^2$

$\varphi = y^2\sin x + xz^3 + f(y,z)$ | $\varphi = y^2\sin x - 4y + f(x,z)$ | $\varphi = xz^3 + f(x,y)$

$$\varphi = y^2\sin x + xz^3 - 4y + c$$

HW S.T $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$ is irrotational. Find $\varphi \ni \vec{F} = \nabla\varphi$

$\nabla\times\vec{F} = 0$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix}$$

$$= \vec{i}(-1+1) - \vec{j}(3z^2 - 3z^2) + \vec{k}(6x - 6x) = 0$$

$\Rightarrow \nabla\times\vec{F}$ is irrotational

To find φ:

$$\vec{F} = \nabla\varphi \Rightarrow (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$$

$$= \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$\Rightarrow \frac{\partial\varphi}{\partial x} = 6xy + z^3$ | $\frac{\partial\varphi}{\partial y} = 3x^2 - z$ | $\frac{\partial\varphi}{\partial z} = 3xz^2 - y$

$\Rightarrow \varphi = 6y\frac{x^2}{2} + z^3x + f(y,z)$ | $\varphi = 3x^2y - zy + f(x,z)$ | $\varphi = 3x\frac{z^3}{3} - yz + f(x,y)$

$\Rightarrow \varphi = 3x^2y + xz^3 + f(y,z)$ | | $\varphi = xz^3 - yz + f(x,y)$

$$\varphi = 3x^2y + xz^3 - yz + c$$

H. Show that the vector $\vec{F} = 3y^4z^2\vec{i} + 4x^3z^2\vec{j} - 3x^2y^2\vec{k}$ is solenoidal.

Given:- $\vec{F} = 3y^4z^2\vec{i} + 4x^3z^2\vec{j} - 3x^2y^2\vec{k}$

To prove: $\text{div}\,\vec{F} = 0$. $\nabla\cdot\vec{F} = 0$

ie) $\text{div}\,\vec{F} = \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(3x^2y^2)$

$= 0 + 0 + 0$

$= 0.$

$\therefore$ $\vec{F}$ is solenoidal.

12) Prove that div $\hat{r} = 2/r$

$$\operatorname{div}\hat{r} = \nabla\cdot\left(\frac{\vec{r}}{r}\right) = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \vec{k}\cdot\right)\cdot\left(\frac{x\vec{i}+y\vec{j}+z\vec{k}}{r}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$= \frac{1}{r} - \frac{1}{r^2}\cdot x\frac{\partial r}{\partial x} + \frac{1}{r} - \frac{1}{r^2}\,y\cdot\frac{\partial r}{\partial y} + \frac{1}{r} - \frac{1}{r^2}\,z\frac{\partial r}{\partial z}$$

$$= \frac{3}{r} - \frac{1}{r^2}\left[x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right]$$

Now $r^2 = x^2+y^2+z^2$

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}\ ;\ \text{lly}\ \frac{\partial r}{\partial y} = \frac{y}{r}\ ;\ \frac{\partial r}{\partial z} = z/r$$

Now $\operatorname{div}\cdot\hat{r} = \frac{3}{r} - \frac{1}{r^2}\left[x\cdot\frac{x}{r} + y\cdot\frac{y}{r} + z\cdot\frac{z}{r}\right]$

$$= \frac{3}{r} - \frac{1}{r^2}\left[\frac{x^2+y^2+z^2}{r}\right] = \frac{3}{r} - \frac{1}{r^2}\cdot\frac{r^2}{r} = \frac{3}{r} - \frac{1}{r}$$

$$\operatorname{div}\hat{r} = \frac{2}{r}$$

13. P.T (curl curl $\vec{F}$) $= \nabla(\operatorname{div}\vec{F}) - \nabla^2\vec{F}$

Given:- $\nabla\times(\nabla\times\vec{F}) = (\nabla\cdot\vec{F})\nabla - (\nabla\cdot\nabla)\vec{F}$

$$\left[\because \vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}\right]$$

$$= \nabla(\nabla\cdot\vec{F}) - \nabla^2\vec{F}$$

$$= \nabla(\operatorname{div}\vec{F}) - \nabla^2\vec{F}$$