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DEPARTMENT OF AEROSPACE ENGINEERING

19ASB304 – COMPUTATIONAL FLUID DYNAMICS FOR AEROSPACE APPLICATIONS

III YEAR VI SEM

UNIT-II FINITE ELEMENT TECHNIQUES

TOPIC: Concept of numerical dissipation

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Numerical Methods

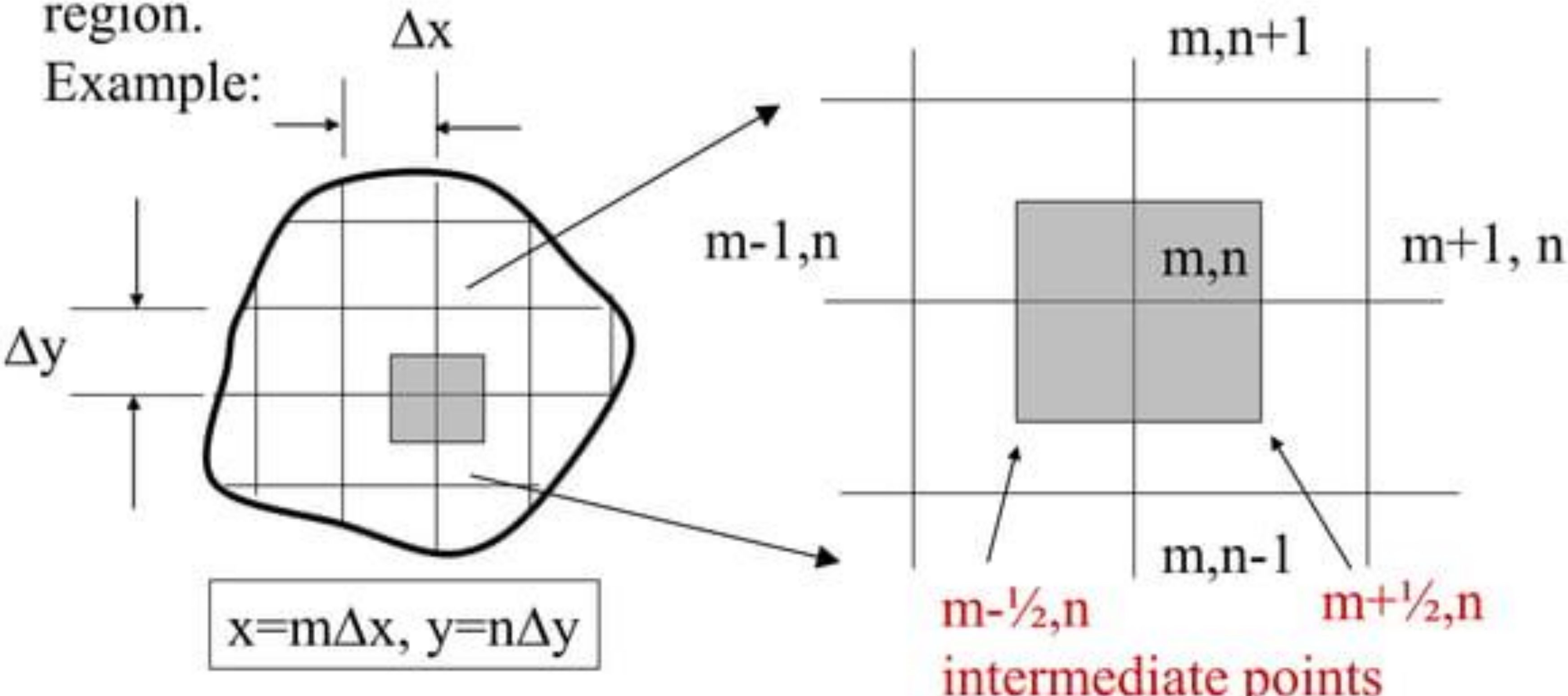
Due to the increasing complexities encountered in the development of modern technology, analytical solutions usually are not available. For these problems, numerical solutions obtained using high-speed computer are very useful, especially when the geometry of the object of interest is irregular, or the boundary conditions are nonlinear. In numerical analysis, two different approaches are commonly used: the finite difference and the finite element methods. In heat transfer problems, the finite difference method is used more often and will be discussed here. The finite difference method involves:

- **Establish nodal networks**
- **Derive finite difference approximations for the governing equation at both interior and exterior nodal points**
- **Develop a system of simultaneous algebraic nodal equations**
- **Solve the system of equations using numerical schemes**

The Nodal Networks

The basic idea is to subdivide the area of interest into sub-volumes with the distance between adjacent nodes by Δx and Δy as shown. If the distance between points is small enough, the differential equation can be approximated locally by a set of finite difference equations. Each node now represents a small region where the nodal temperature is a measure of the average temperature of the region.

Example:



Finite Difference Approximation

Heat Diffusion Equation: $\nabla^2 T + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$,

where $\alpha = \frac{\mathrm{k}}{\rho \mathrm{C_P} V}$ is the thermal diffusivity

No generation and steady state: $\dot{\mathrm{q}}=0$ and $\frac{\partial}{\partial \mathrm{t}} = 0, \Rightarrow \nabla^2 T = 0$

First, approximated the first order differentiation at intermediate points (m+1/2,n) & (m-1/2,n)

$$\left.\frac{\partial \mathrm{T}}{\partial \mathrm{x}}\right|_{(m+1/2,n)} \approx \left.\frac{\Delta T}{\Delta x}\right|_{(m+1/2,n)} = \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$\left.\frac{\partial \mathrm{T}}{\partial \mathrm{x}}\right|_{(m-1/2,n)} \approx \left.\frac{\Delta T}{\Delta x}\right|_{(m-1/2,n)} = \frac{T_{m,n} - T_{m-1,n}}{\Delta x}$$

Finite Difference Approximation (cont.)

Next, approximate the second order differentiation at m,n

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} \approx \frac{\left.\partial T/\partial x\right|_{m+1/2,n} - \left.\partial T/\partial x\right|_{m-1/2,n}}{\Delta x}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} \approx \frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2}$$

Similarly, the approximation can be applied to the other dimension y

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{m,n} \approx \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2}$$

Finite Difference Approximation (cont.)

$$\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)_{m,n} \approx \frac{T_{m+1,n}+T_{m-1,n}-2T_{m,n}}{(\Delta x)^2}+\frac{T_{m,n+1}+T_{m,n-1}-2T_{m,n}}{(\Delta y)^2}$$

To model the steady state, no generation heat equation: $\nabla^2 T = 0$

This approximation can be simplified by specify $\Delta x = \Delta y$

and the nodal equation can be obtained as

$$T_{m+1,n}+T_{m-1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}=0$$

This equation approximates the nodal temperature distribution based on the heat equation. This approximation is improved when the distance between the adjacent nodal points is decreased:

Since $\lim(\Delta x \rightarrow 0)\dfrac{\Delta T}{\Delta x}=\dfrac{\partial T}{\partial x}, \lim(\Delta y \rightarrow 0)\dfrac{\Delta T}{\Delta y}=\dfrac{\partial T}{\partial y}$

A System of Algebraic Equations

• The nodal equations derived previously are valid for all interior points satisfying the steady state, no generation heat equation. For each node, there is one such equation.
For example: for nodal point m=3, n=4, the equation is
$T_{2,4} + T_{4,4} + T_{3,3} + T_{3,5} - 4T_{3,4} = 0$
$T_{3,4} = (1/4)(T_{2,4} + T_{4,4} + T_{3,3} + T_{3,5})$

• Nodal relation table for exterior nodes (boundary conditions) can be found in standard heat transfer textbooks. (ex. F.P. Incropera & D.P. DeWitt, "Introduction to Heat Transfer".)

• Derive one equation for each nodal point (including both interior and exterior points) in the system of interest. The result is a system of N algebraic equations for a total of N nodal points.

Matrix Form

The system of equations:

$$
\begin{aligned}
&a_{11}T_1 + a_{12}T_2 + \cdots + a_{1N}T_N = C_1 \\
&a_{21}T_1 + a_{22}T_2 + \cdots + a_{2N}T_N = C_2 \\
&\vdots \qquad\quad \vdots \qquad\quad \vdots \quad \vdots \qquad\quad \vdots \\
&a_{N1}T_1 + a_{N2}T_2 + \cdots + a_{NN}T_N = C_N
\end{aligned}
$$

A total of N algebraic equations for the N nodal points and the system can be expressed as a matrix formulation: [**A**][**T**]=[**C**]

$$
\textit{where } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}, T = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{bmatrix}, C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix}
$$

Numerical Solutions

Matrix form: [**A**][**T**]=[**C**].
From linear algebra: $[\mathbf{A}]^{-1}[\mathbf{A}][\mathbf{T}]=[\mathbf{A}]^{-1}[\mathbf{C}]$, $[\mathbf{T}]=[\mathbf{A}]^{-1}[\mathbf{C}]$
where $[\mathbf{A}]^{-1}$ is the inverse of matrix [**A**]. [**T**] is the solution vector.

- Matrix inversion requires cumbersome numerical computations and is not efficient if the order of the matrix is high (>10)

- Gauss elimination method and other matrix solvers are usually available in many numerical solution package. For example, "Numerical Recipes" by Cambridge University Press or their web source at www.nr.com.

- For high order matrix, iterative methods are usually more efficient. The famous Jacobi & Gauss-Seidel iteration methods will be introduced in the following.

Iteration

General algebraic equation for nodal point:

$$\sum_{j=1}^{i-1} a_{ij} T_j + a_{ii} T_i + \sum_{j=i+1}^{N} a_{ij} T_j = C_i,$$

$$(\text{Example}: a_{31} T_1 + a_{32} T_2 + a_{33} T_3 + \cdots + a_{1N} T_N = C_1,\ i = 3)$$

Rewrite the equation of the form:

Replace (k) by (k-1) for the Jacobi iteration

$$T_i^{(k)} = \frac{C_i}{a_{ii}} - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} T_j^{(k)} - \sum_{j=i+1}^{N} \frac{a_{ij}}{a_{ii}} T_j^{(k-1)}$$

- (k) - specify the level of the iteration, (k-1) means the present level and (k) represents the new level.
- An initial guess (k=0) is needed to start the iteration.
- By substituting iterated values at (k-1) into the equation, the new values at iteration (k) can be estimated
- The iteration will be stopped when $\max \left| T_i^{(k)} - T_i^{(k-1)} \right| \leq \varepsilon$, where ε specifies a predetermined value of acceptable error

Example

Solve the following system of equations using (a) the Jacobi methos, (b) the Gauss Seidel iteration method.

$$\begin{bmatrix} 4 & 2 & 1 \\ -1 & 2 & 0 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 11 \\ 3 \\ 16 \end{bmatrix}$$

$$4X + 2Y + Z = 11,$$

$$-X + 2Y + 0 * Z = 3,$$

$$2X + Y + 4Z = 16$$

Reorganize into new form:

$$X = \frac{11}{4} - \frac{1}{2} Y - \frac{1}{4} Z$$

$$Y = \frac{3}{2} + \frac{1}{2} X + 0 * Z$$

$$Z = 4 - \frac{1}{2} X - \frac{1}{4} Y$$

(a) Jacobi method: use initial guess $X^0 = Y^0 = Z^0 = 1$,

stop when $\max \left| X^k - X^{k-1}, Y^k - Y^{k-1}, Z^k - Z^{k-1} \right| \leq 0.1$

First iteration:

$X^1 = (11/4) - (1/2)Y^0 - (1/4)Z^0 = 2$

$Y^1 = (3/2) + (1/2)X^0 = 2$

$Z^1 = 4 - (1/2)X^0 - (1/4)Y^0 = 13/4$

Example (cont.)

Second iteration: use the iterated values $X^1=2$, $Y^1=2$, $Z^1=13/4$

$X^2 = (11/4) - (1/2)Y^1 - (1/4)Z^1 = 15/16$

$Y^2 = (3/2) + (1/2)X^1 = 5/2$

$Z^2 = 4 - (1/2)X^1 - (1/4)Y^1 = 5/2$

Converging Process:

$$[1,1,1], \left[2,2,\frac{13}{4}\right], \left[\frac{15}{16},\frac{5}{2},\frac{5}{2}\right], \left[\frac{7}{8},\frac{63}{32},\frac{93}{32}\right], \left[\frac{133}{128},\frac{31}{16},\frac{393}{128}\right]$$

$\left[\frac{519}{512},\frac{517}{256},\frac{767}{256}\right]$. Stop the iteration when

$$\max \left|X^5 - X^4, Y^5 - Y^4, Z^5 - Z^4\right| \le 0.1$$

Final solution [1.014, 2.02, 2.996]

Exact solution [1, 2, 3]

Example (cont.)

(b) Gauss-Seidel iteration: Substitute the iterated values into the iterative process immediately after they are computed.

Use initial guess $X^0 = Y^0 = Z^0 = 1$

$$X = \frac{11}{4} - \frac{1}{2}Y - \frac{1}{4}Z,\ Y = \frac{3}{2} + \frac{1}{2}X,\ Z = 4 - \frac{1}{2}X - \frac{1}{4}Y$$

First iteration: $X^1 = \frac{11}{4} - \frac{1}{2}(Y^0) - \frac{1}{4}(Z^0) = 2$ Immediate substitution

$$Y^1 = \frac{3}{2} + \frac{1}{2}X^1 = \frac{3}{2} + \frac{1}{2}(2) = \frac{5}{2}$$

$$Z^1 = 4 - \frac{1}{2}X^1 - \frac{1}{4}Y^1 = 4 - \frac{1}{2}(2) - \frac{1}{4}\left(\frac{5}{2}\right) = \frac{19}{8}$$

Converging process: $[1,1,1]$, $\left[2, \frac{5}{2}, \frac{19}{8}\right]$, $\left[\frac{29}{32}, \frac{125}{64}, \frac{783}{256}\right]$, $\left[\frac{1033}{1024}, \frac{4095}{2048}, \frac{24541}{8192}\right]$

The iterated solution [1.009, 1.9995, 2.996] and it converges faster