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DEPARTMENT OF AEROSPACE ENGINEERING

19ASB304 – COMPUTATIONAL FLUID DYNAMICS FOR AEROSPACE APPLICATIONS
III YEAR VI SEM
UNIT-II FINITE ELEMENT TECHNIQUES
TOPIC: Stability properties of explicit and implicit methods

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Explicit Method

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \right]$$

- Explicit method uses the fact that we know the dependent variable, u at all x at time t from initial conditions
- Since the equation contains only one unknown, u_i^{n+1} (i.e. u at time t+Δt), it can be obtained directly from known values of u at t
- The solution takes the form of a "marching" procedure in steps of time

Crank – Nicolson Implicit Method

- The unknown value u at time level (n+1) is expressed both in terms of known quantities at n and unknown quantities at (n+1).
- The spatial differences on RHS are expressed in terms of averages between time level n and (n+1) :

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\alpha}{2}\left[\frac{u_{i+1}^{n+1} + u_{i+1}^n - 2u_i^{n+1} - 2u_i^n + u_{i-1}^{n+1} + u_{i-1}^n}{\Delta x^2}\right]$$

- The above equation cannot result in a solution of u_i^{n+1} at grid point i.
- The eq. is written at all grid points resulting in a system of algebraic equations which can be solved simultaneously for u at all i at time level (n+1).

Crank – Nicolson Implicit Method

- The equation can be rearranged as

$$-u_{i-1}^{n+1} + \frac{2+2r}{r} u_i^{n+1} - u_{i+1}^{n+1} = u_{i-1}^{n} + \frac{2-2r}{r} u_i^{n} + u_{i+1}^{n}$$

where $r = \alpha\Delta t/(\Delta x)^2$

- On application of eq. at all grid points from i=1 to i=k+1 , the system of eqs. with boundary conditions u=A at x=0 and u=D at x=L can be expressed in the form of Ax = C

$$\begin{bmatrix} B(1) & -1 & 0 & 0 & \dots & 0 \\ -1 & B(2) & -1 & 0 & \dots & 0 \\ 0 & -1 & B(3) & -1 & \dots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & -1 & B(k-1) \end{bmatrix} \begin{bmatrix} u_2^{n+1} \\ u_3^{n+1} \\ u_4^{n+1} \\ \vdots \\ u_k^{n+1} \end{bmatrix} = \begin{bmatrix} (C(1)+A)^n \\ C(2)^n \\ C(3)^n \\ \vdots \\ (C(k-1)+D)^n \end{bmatrix}$$

- A is the tridiagonal coefficient matrix and x is the solution vector. The eq. can be solved using Thomas Algorithm

Explicit ~ Implicit – A Comparison

- Explicit Method
  - Easy to set up.
  - Constraint on mesh width, time-step.
  - Less computer time.
- Implicit Method
  - Complicated to set up.
  - Larger computer time.
  - No constraint on time step.
  - Can be solved using Thomas Algorithm.

Consistency

- A finite difference representation of a PDE is said to be consistent if:

$$\lim_{mesh\to 0}(PDE - FDE) = \lim_{mesh\to 0}(TE) = 0$$

- For equations where truncation error is $\mathcal{O}(\Delta x)$ or $\mathcal{O}(\Delta t)$ or higher orders, TE vanishes as the mesh is refined
- However, for schemes where TE is $\mathcal{O}(\Delta t/\Delta x)$, the scheme is not consistent unless mesh is refined in a manner such that $\Delta t/\Delta x \to 0$
- For the Dufort-Frankel differencing scheme (1953), if $\Delta t/\Delta x$ does not tend to zero, a parabolic PDE may end up as a hyperbolic equation

Convergence

- A solution of the algebraic equations that approximate a PDE is convergent if the approximate solution approaches the exact solution of the PDE for each value of the independent variable as the grid spacing tends to zero :

$$u_i^n = \bar{u}(x_i, t_n) \; as \; \Delta x, \Delta t \to 0$$

RHS is the solution of algebraic equation

Errors & Stability Analysis

- Errors :
  - A = Analytical solution of PDE
  - D = Exact solution of finite difference equation
  - N = Numerical solution from a real computer with finite accuracy
  - Discretization Error = A –D = Truncation error + error introduced due to the treatment of boundary condition
  - Round-off Error = ε= N –D

$$N = \varepsilon + D$$

ε will be referred to as "error" henceforth

Errors & Stability Analysis

- Consider the 1-D unsteady state heat conduction equation and its FDE :

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \quad \longrightarrow \quad \frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \right]$$

- N must satisfy the finite difference equation :

$$\frac{D_i^{n+1} + \varepsilon_i^{n+1} - D_i^n - \varepsilon_i^n}{\Delta t} = \alpha \left[\frac{D_{i+1}^n + \varepsilon_{i+1}^n - 2D_i^n - 2\varepsilon_i^n + D_{i-1}^n + \varepsilon_{i-1}^n}{\Delta x^2} \right]$$

- Also, D being the exact solution also satisfies FDE :

$$\frac{D_i^{n+1} - D_i^n}{\Delta t} = \alpha \left[\frac{D_{i+1}^n - 2D_i^n + D_{i-1}^n}{\Delta x^2} \right]$$

- Subtracting above 2 equations, we see that error ε also satisfies FDE :

$$\frac{\varepsilon_i^{n+1} - \varepsilon_i^n}{\Delta t} = \alpha \left[\frac{\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n}{\Delta x^2} \right]$$

- If errors ε_i's shrink or remain same from step n to n+1, solution is stable. Condition for stability is :

$$\left| \frac{\varepsilon_i^{n+1}}{\varepsilon_i^n} \right| \leq 1$$

Application in Fluid Flow Equations

Introduction

- Fluid mechanics: More complex, governing PDE's form a nonlinear system.
- Burger's Equation: $\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} = \nu\frac{\partial^2 \zeta}{\partial x^2}$ => Includes time dependent, convective and diffusive term.
- Here 'u': velocity, 'γ': coefficient of viscosity, & 'ζ': any property which can be transported or diffused.
- Neglecting viscous term, remaining equation is a simple analog of Euler's equation : $\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} = 0$

Conservative Property

- FDE possesses conservative property if it preserves integral conservation relations of the continuum
- Consider *Vorticity Transport Equation:* $\frac{\partial \omega}{\partial t} = -(V.\nabla)\omega + \nu\nabla^2\omega$

  where ∇ is nabla, V is fluid velocity and ω is vorticity.
- Integrating over a fixed region we get,

$$\int_{\Re} \frac{\partial \omega}{\partial t} d\Re = -\int_{\Re} (V \cdot \nabla)\omega d\Re + \int_{\Re} \nu \nabla^2 \omega d\Re$$

  which can be written as :

$$\frac{\partial}{\partial t}\int_{\Re} \omega \, d\Re = -\int_{A_o} (V\omega) \cdot n \, dA + \nu \int_{A_o} (\nabla \omega) \cdot n \, dA$$

  i.e. rate of accumulation of ω in $\Re$ is equal to net advective flux rate plus net diffusive flux rate of ω across A_o into $\Re$
- The concept of conservative property is to maintain this integral relation in finite difference representation.

Conservative Property

- Consider inviscid Burger's equation :

$$\frac{\partial \omega}{\partial t} = -\frac{\partial}{\partial x}(u\omega) \quad \xrightarrow{\text{FDE Analog}} \quad \frac{\omega_i^{n+1} - \omega_i^n}{\Delta t} = -\frac{u_{i+1}^n \omega_{i+1}^n - u_{i-1}^n \omega_{i-1}^n}{2\Delta x}$$

- Evaluating the integral $\frac{1}{\Delta t}\sum_{i=I_1}^{I_2} \omega \Delta x$ over a region running from $i=I_1$ to $i=I_2$:

$$\frac{1}{\Delta t}\left[\sum_{i=I_1}^{I_2} \omega_i^{n+1}\Delta x - \sum_{i=I_1}^{I_2} \omega_i^n \Delta x\right] = (u\omega)_{I_1-\frac{1}{2}} - (u\omega)_{I_2+\frac{1}{2}}$$

Thus, the FDE analogous to inviscid part of the integral has preserved the conservative property.

- For non-conservative form of inviscid Burger's equation: $\frac{\partial \omega}{\partial t} = -u\frac{\partial \omega}{\partial x}$

$$\frac{1}{\Delta t}\left[\sum_{i=I_1}^{I_2} \omega_i^{n+1}\Delta x - \sum_{i=I_1}^{I_2} \omega_i^n \Delta x\right] = \frac{1}{2}\sum_{i=I_1}^{I_2} [u_i^n \omega_{i-1}^n - u_i^n \omega_{i+1}^n]$$

i.e. FDE analog has failed to preserve the conservative property

Transportive Property

- FDE formulation of a flow is said to possess the transportive property if the effect of perturbation is convected only in the direction of velocity
- Consider model Burger's equation in conservative form and a perturbation $\varepsilon_m = \delta$ in ζ for u>0, all other ε=0
- Using FTCS, we find the transportive property to be violated
- On the contrary when an upwind scheme is used,

$$\frac{\zeta_i^{n+1} - \zeta_i^n}{\Delta t} = -\frac{u\,\zeta_i^n - u\,\zeta_{i-1}^n}{\Delta x}$$

$$\frac{\zeta_{m+1}^{n+1} - \zeta_{m+1}^n}{\Delta t} = +\frac{u\delta}{\Delta x}$$ => Downstream Location (m+1)

$$\frac{\zeta_{m+1}^{n+1} - \zeta_{m+1}^n}{\Delta t} = -\frac{u\delta}{\Delta x}$$ => Point m of disturbance

$$\frac{\zeta_{m+1}^{n+1} - \zeta_{m+1}^n}{\Delta t} = 0$$ => Upstream Location (m-1)

- Upwind method maintains unidirectional flow of information.

The Upwind Method

- The inviscid Burger's equation in the following forms are unconditionally unstable :

$$\frac{\zeta_i^{n+1} - \zeta_i^n}{\Delta t} + u\,\frac{\zeta_{i+1}^n - \zeta_i^n}{\Delta x} = 0$$

$$\frac{\zeta_i^{n+1} - \zeta_i^n}{\Delta t} + u\,\frac{\zeta_{i+1}^n - \zeta_{i-1}^n}{2\,\Delta x} = 0$$

- The equations can be made stable by using backward space difference scheme if u > 0 and forward space difference scheme if u < 0 :

$$\frac{\zeta_i^{n+1} - \zeta_i^n}{\Delta t} = -\frac{u\,\zeta_i^n - u\,\zeta_{i-1}^n}{\Delta x} + \text{viscous term, for } u > 0$$

$$\frac{\zeta_i^{n+1} - \zeta_i^n}{\Delta t} = -\frac{u\,\zeta_{i+1}^n - u\,\zeta_i^n}{\Delta x} + \text{viscous term, for } u < 0$$

- Upwind method of discretization is necessary in convection dominated flows.

THANK YOU