Stoke's Theorem:

If $\vec{F}$ is any continuous differentiable vector function and S is a surface enclosed by a curve C then,

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\, ds$$

where $\hat{n}$ is the unit normal vector at any point of S.

Note:

1. If $\vec{F}$ is irrotational, $\nabla\times\vec{F} = 0$.

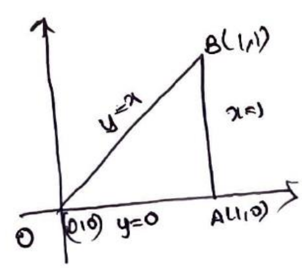
$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\, ds = 0$$ & Hence $\vec{F}$ is conservative

2. Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

$$\int_C P\,dx + Q\,dy + R\,dz = \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dz\,dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

Problems:-

(i) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ by stoke's theorem where $\vec{F} = y^2\vec{i} + x^2\vec{j} - (x+z)\vec{k}$ and C is the boundary of the triangle with vertices at (0,0,0), (1,0,0), (1,1,0)



By stoke's theorem

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\, ds$$

Since z coordinate is zero in all the three vertices of the given triangle, the triangle lies on the xy plane

$$\vec{F} = y^2\vec{i} + x^2\vec{j} - (x+2)\vec{k}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+2) \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(-1-0) + \vec{k}(2x-2y)$$

$$= \vec{j} + 2(x-y)\vec{k}$$

Since the triangle lies on XOY plane and hence the unit vector normal to the surface OAB is $\vec{k}$.

ie) $\hat{n} = \vec{k}$

$$\nabla\times\vec{F}\cdot\hat{n} = [\vec{j} + 2(x-y)\vec{k}]\cdot\vec{k} = 2(x-y)$$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds = \iint_S 2(x-y)\,dx\,dy$$

($\because$ S lies on xy plane $ds = dx\,dy$)

$$= 2\int_0^1\int_y^1 (x-y)\,dx\,dy$$

[$x \to y$ to 1
$y \to 0$ to 1]

$$= 2\int_0^1 \left[\frac{x^2}{2} - xy\right]_y^1 dy$$

$$= 2\int_0^1 \left[\frac{1}{2} - y - \frac{y^2}{2} + y^2\right] dy$$

$$= 2\left[\frac{1}{2}y - \frac{y^2}{2} - \frac{y^3}{6} + \frac{y^3}{3}\right]_0^1$$

$$= 2\left(\frac{1}{2} - \frac{1}{2} - \frac{1}{6} + \frac{1}{3}\right) = \frac{1}{3}.$$

2. Verify Stoke's theorem for a vector field defined by $\vec{F} = (x^2-y^2)\vec{i} + 2xy\vec{j}$ in the rectangular region in the xoy plane bounded by the lines $x=0$, $x=a$, $y=0$ and $y=b$

By stoke's theorem,

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\, ds$$

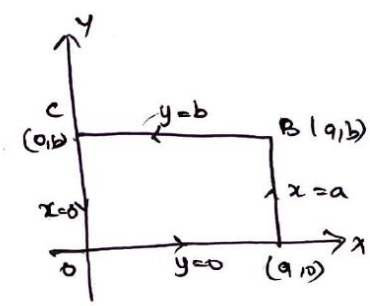
RHS: Given $\vec{F} = (x^2-y^2)\vec{i} + 2xy\vec{j}$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2 & 2xy & 0 \end{vmatrix} = \vec{i}(0) - \vec{j}(0) + \vec{k}(2y+2y)$$

$$= 4y\vec{k}$$

Here the surface S denotes the rectangle OABC and the unit outward normal vector is $\vec{k}$

i.e.) $\hat{n} = \vec{k}$



$\therefore$ curl $\vec{F}\cdot\hat{n}\, ds$

$= 4y\vec{k}\cdot\vec{k}\ dxdy$

$= 4y\ dxdy$.

$$\iint \nabla\times\vec{F}\cdot\hat{n}\, ds = \iint_S 4y\, dxdy = \int_0^b\int_0^a 4y\, dxdy$$

$$= 4\int_0^b [x]_0^a y\, dy$$

$$= 4a\left[\frac{y^2}{2}\right] = \frac{4ab^2}{2}$$

$$= 2ab^2 \rightarrow (1)$$