Gauss Divergence Theorem:

If $\vec{F}$ is a vector point function, finite and differentiable in a region R bounded by a closed surface S, then the surface integral of the normal component of $\vec{F}$ taken over S is equal to the integral of divergence of $\vec{F}$ taken over V.

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv.$$

where $\hat{n}$ is the unit vector in the +ve (outward drawn) normal to S.

1. Verify Gauss divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ taken over the rectangular parallelopiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.

By Gauss divergence theorem,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv.$$

RHS: $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$

$$\nabla\cdot\vec{F} = (2x) + (2y) + (2z)$$

$$= 2(x + y + z)$$

$$\iiint_V \nabla\cdot\vec{F}\,dv = \int_0^a\int_0^b\int_0^c 2(x+y+z)\,dz\,dy\,dx$$

$$= 2\int_0^a\int_0^b \left[xz + yz + \frac{z^2}{2}\right]_0^c dy\,dx$$

$$= 2\int_0^a\int_0^b \left[cx + cy + \frac{c^2}{2}\right] dy\,dx$$

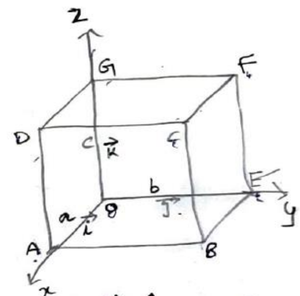
$$= 2\int_0^a \left[cxy + \frac{cy^2}{2} + \frac{c^2}{2}y\right]_0^b dx$$

$$= 2\int_0^a \left[bcx + \frac{b^2c}{2} + \frac{bc^2}{2}\right] dx$$

$$= 2\left[\frac{bcx^2}{2} + \frac{b^2cx}{2} + \frac{bc^2x}{2}\right]_0^a$$

$$= 2\left[\frac{a^2bc + ab^2c + abc^2}{2}\right]$$

$$= abc\,[a+b+c] \longrightarrow (1).$$



LHS : $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$

Surface	$\hat{n}$	$\vec{F}\cdot\hat{n}$	ds	Face eqn	$\vec{F}\cdot\hat{n}$ on S.
S_1 - ABCD	$\vec{i}$	$x^2 - yz$	$dy\,dz$	$x = a$	$a^2 - yz$
S_2 - OEFG	$-\vec{i}$	$yz - x^2$	$dy\,dz$	$x = 0$	yz
S_3 - BCEF	$\vec{j}$	$y^2 - zx$	$dx\,dz$	$y = b$	$b^2 - zx$
S_4 - OADG	$-\vec{j}$	$zx - y^2$	$dx\,dz$	$y = 0$	zx
S_5 - DCGF	$\vec{k}$	$z^2 - xy$	$dx\,dy$	$z = c$	$c^2 - xy$
S_6 - OABE	$-\vec{k}$	$xy - z^2$	$dx\,dy$	$z = 0$	xy

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^b (a^2 - yz)\,dy\,dz = \int_0^c \left(a^2y - \frac{y^2}{2}z\right)_0^b dz$$

$$= \int_0^c \left(a^2b - \frac{b^2z}{2}\right) dz = \left(a^2bz - \frac{b^2z^2}{4}\right)_0^c$$

$$= a^2bc - \frac{b^2c^2}{4}$$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^b yz\,dy\,dz = \int_0^c \left[\frac{y^2}{2}z\right]_0^b dz$$

$$= \int_0^c \left(\frac{b^2}{2}z\right)dz = \left(\frac{b^2z^2}{4}\right)_0^c = \frac{b^2c^2}{4}$$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^a (b^2 - zx)\,dx\,dz = \int_0^c \left(b^2x - \frac{zx^2}{2}\right)_0^a dz$$

$$= \int_0^c \left(b^2a - \frac{za^2}{2}\right)dz = \left(b^2az - \frac{z^2a^2}{4}\right)_0^c$$

$$= ab^2c - \frac{a^2c^2}{4}$$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^a zx\,dx\,dz = \int_0^c \left(\frac{zx^2}{2}\right)_0^a dz = \int_0^c \frac{a^2z}{2}\,dz$$

$$= \left[\frac{a^2z^2}{4}\right]_0^c = \frac{a^2c^2}{4}$$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \int_0^b\int_0^a (c^2 - xy)\,dx\,dy = \int_0^b \left(c^2x - \frac{x^2y}{2}\right)_0^a dy$$

$$= \int_0^b \left(c^2a - \frac{a^2y}{2}\right)dy = \left(ac^2y - \frac{a^2y^2}{4}\right)_0^b$$

$$= abc^2 - \frac{a^2b^2}{4}$$

$$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \int_0^b\int_0^a xy\,dx\,dy = \int_0^b \left(\frac{x^2y}{2}\right)_0^a dy = \int_0^b \left(\frac{a^2y}{2}\right)dy$$

$$= \left(\frac{a^2y^2}{4}\right)_0^b = \frac{a^2b^2}{4}.$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = a^2bc - \cancel{\frac{b^2c^2}{4}} + \cancel{\frac{b^2c^2}{4}} + ab^2c - \cancel{\frac{a^2c^2}{4}} + \cancel{\frac{a^2c^2}{4}}$$

$$+ abc^2 - \cancel{\frac{a^2b^2}{4}} + \cancel{\frac{a^2b^2}{4}}$$

$$= a^2bc + ab^2c + abc^2$$

$$= abc(a+b+c) \rightarrow ②.$$

from ① & ②

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv.$$

2. Verify divergence theorem for $\vec{F} = (2x-z)\vec{i} + x^2y\vec{j} - xz^2\vec{k}$ over the cube bounded by $x=0, x=1, y=0, y=1, z=0, z=1$

By Gauss divergence theorem,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

RHS $\Rightarrow$ $\vec{F} = (2x-z)\vec{i} + x^2y\vec{j} - xz^2\vec{k}$

$$\nabla\cdot\vec{F} = 2 + x^2 - 2xz$$

$$\iiint_V \nabla\cdot\vec{F}\,dv = \int_0^1\int_0^1\int_0^1 2+x^2-2xz\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 \left(2x + \frac{x^3}{3} - \frac{2x^2}{2}z\right)_0^1 dy\,dz$$

$$= \int_0^1\int_0^1 (2 + 1/3 - z)\,dy\,dz = \int_0^1 \left[2y + \frac{y}{3} - \frac{zy}{1}\right]_0^1 dz$$

$$= \int_0^1 \left(2+\frac{1}{3} - z\right)dz = \left(\frac{7}{3}z - \frac{z^2}{2}\right)_0^1$$

$$= \frac{7}{3} - \frac{1}{2} = \frac{14-3}{6} = \frac{11}{6}$$

$$\iiint_V \nabla\cdot\vec{F}\,dv = \frac{11}{6} \longrightarrow ①$$

LHS $\Rightarrow \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$

Surface	$\hat{n}$	$\vec{F}\cdot\hat{n}$	ds	Face eqn	$\vec{F}\cdot\hat{n}$ on S
S_1-ABCD	$\vec{i}$	$2x-z$	$dy\,dz$	$x=1$	$2-z$
S_2-OEFG	$-\vec{i}$	$z-2x$	$dy\,dz$	$x=0$	z
S_3-BCFE	$\vec{j}$	x^2y	$dx\,dz$	$y=1$	x^2
S_4-OADG	$-\vec{j}$	$-x^2y$	$dx\,dz$	$y=0$	0
S_5-DCFG	$\vec{k}$	$-x^2z$	$dx\,dy$	$z=1$	$-x$
S_6-OABE	$-\vec{k}$	x^2z	$dx\,dy$	$z=0$	0

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (2-z)\,dy\,dz = \int_0^1 (2y-yz)_0^1\,dz = \int_0^1 (2-z)\,dz$$

$$= \left[2z - \frac{z^2}{2}\right]_0^1 = 2-\frac{1}{2} = \frac{3}{2}.$$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 z\,dy\,dz = \int_0^1 (zy)_0^1\,dz = \int_0^1 z\,dz = \left[\frac{z^2}{2}\right]_0^1 = \frac{1}{2}$$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 x^2\,dx\,dz = \int_0^1 \left(\frac{x^3}{3}\right)_0^1 dz = \int_0^1 \frac{1}{3}\,dz = \left[\frac{z}{3}\right]_0^1 = \frac{1}{3}$$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 -x\,dx\,dy = \int_0^1 \left(\frac{-x^2}{2}\right)_0^1 dy = \int_0^1 \left(\frac{-1}{2}\right)dy = \left[\frac{-y}{2}\right]_0^1 = \frac{-1}{2}$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \frac{3}{2} + \frac{1}{2} + \frac{1}{3} - \frac{1}{2} = \frac{3}{2} + \frac{1}{3}$$

$$= \frac{9+2}{6} = \frac{11}{6} \rightarrow (2)$$

From (1) & (2)

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

Q. Verify divergence theorem for $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$ over the cube formed by $x = \pm 1$, $y = \pm 1$, $z = \pm 1$

By gauss divergence theorem,

$$\iint_S \vec{F}.\hat{n}\, ds = \iiint_V \nabla.\vec{F}\, dv$$

RHS $\Rightarrow$ $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$

$\nabla.\vec{F} = 2x + y$

$$\iiint_V \nabla.\vec{F}\, dv = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (2x+y)\, dz\, dy\, dx$$

$$= \int_{-1}^{1}\int_{-1}^{1} \left[2xz + yz\right]_{-1}^{1} dy\, dx = \int_{-1}^{1}\int_{-1}^{1} \left[(2x+y) - (-2x-y)\right] dy\, dx$$

$$= 2\int_{-1}^{1}\int_{-1}^{1} (2x+y)\, dy\, dx = 2\int_{-1}^{1} \left[2xy + \frac{y^2}{2}\right]_{-1}^{1} dx$$

$$= 2\int_{-1}^{1} \left[\left(2x + 1/2\right) - \left(-2x + 1/2\right)\right] dx = 2\int_{-1}^{1} 4x\, dx$$

$$= 8\left[\frac{x^2}{2}\right]_{-1}^{1} = 4(1-1) = 0. \rightarrow (1)$$

LHS $\Rightarrow$ $\iint_S \vec{F}.\hat{n}\, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$

Surface	$\hat{n}$	$\vec{F}.\hat{n}$	ds	Face eqn	$\vec{F}.\hat{n}$ on ds
S_1 - ABCD	$\vec{i}$	x^2	$dy\,dz$	$x=1$	1
S_2 - OEFG	$-\vec{i}$	$-x^2$	$dy\,dz$	$x=-1$	-1
S_3 - BCFE	$\vec{j}$	z	$dx\,dz$	$y=1$	z
S_4 - OADG	$-\vec{j}$	$-z$	$dx\,dz$	$y=-1$	$-z$
S_5 - DCFG	$\vec{k}$	yz	$dx\,dy$	$z=1$	y
S_6 - OABE	$-\vec{k}$	$-yz$	$dx\,dy$	$z=-1$	y

$$\iint_{S_1} \vec{F}.\hat{n}\, ds = \int_{-1}^{1}\int_{-1}^{1} dy\,dz = \int_{-1}^{1} [y]_{-1}^{1}\, dz = \int_{-1}^{1} (1+1)\, dz = 2\int_{-1}^{1} dz = 4$$

$$\iint_{S_2} \vec{F}.\hat{n}\, ds = -\int_{-1}^{1}\int_{-1}^{1} dy\,dz = -\int_{-1}^{1} (y)_{-1}^{1} dz = -2\int_{-1}^{1} dz = -2[z]_{-1}^{1} = -4$$

$$\iint_{S_3} \vec{F}.\hat{n}\, ds = \int_{-1}^{1}\int_{-1}^{1} z\, dx\,dz = \int_{-1}^{1} (zx)_{-1}^{1} dz = 2\int_{-1}^{1} z\,dz = 2\left[\frac{z^2}{2}\right]_{-1}^{1} = 0$$

$$\iint_{S_4} \vec{F}.\hat{n}\, ds = \int_{-1}^{1}\int_{-1}^{1} -z\, dx\,dz = \int_{-1}^{1} (-zx)_{-1}^{1} dz = -2\int_{-1}^{1} (z)\,dz = 0$$

$$\iint_{S_5} \vec{F}.\hat{n}\, ds = \int_{-1}^{1}\int_{-1}^{1} y\, dx\,dy = 2\int_{-1}^{1} y\,dy = 2\left(\frac{y^2}{2}\right)_{-1}^{1} = 0.$$

$$\iint_{S_6} \vec{F}.\hat{n}\, ds = \int_{-1}^{1}\int_{-1}^{1} y\, dx\,dy = 0.$$

$$\iint_{S} \vec{F}.\hat{n}\, ds = 4 - 4 + 0 + 0 + 0 + 0 = 0 \rightarrow (2)$$

From (1) & (2)

$$\iint_{S} \vec{F}.\hat{n}\, ds = \iiint_{V} \nabla.\vec{F}\, dv.$$

4. Verify Gauss divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube $x=0, x=1, y=0, y=1, z=0, z=1$

By Gauss divergence theorem.

$$\iint_{S} \vec{F}.\hat{n}\, ds = \iiint_{V} \nabla.\vec{F}\, dv$$

RHS ⇒

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}\vec{F_1} + \frac{\partial \vec{F_2}}{\partial y} + \frac{\partial \vec{F_3}}{\partial z}$$

$$= 4z - 2y + y$$

$$= 4z - y$$

$$\iiint_V \nabla\cdot\vec{F}\,dv = \int_0^1\int_0^1\int_0^1 (4z-y)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 \left[4xz - yx\right]_0^1 dy\,dz = \int_0^1\int_0^1 (4z-y)\,dy\,dz$$

$$= \int_0^1 \left(4zy - y^2/2\right)_0^1 dz = \int_0^1 \left[4z - 1/2\right] dz$$

$$= \left[\frac{4z^2}{2} - \frac{1}{2}z\right]_0^1 = \left[2z^2 - \frac{1}{2}z\right]_0^1$$

$$= \left[2 - 1/2\right] = 3/2. \rightarrow (1).$$

LHS ⇒

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

Surface	$\hat{n}$	$\vec{F}\cdot\hat{n}$	ds	Face eqn	$\vec{F}\cdot\hat{n}$ on S
S_1 - ABCD	$\vec{i}$	$4xz$	$dy\,dz$	$x=1$	$4z$
S_2 - OEFG	$-\vec{i}$	$-4xz$	$dy\,dz$	$x=0$	0
S_3 - BCFE	$\vec{j}$	$-y^2$	$dx\,dz$	$y=1$	-1
S_4 - OADG	$-\vec{j}$	y^2	$dx\,dz$	$y=0$	0
S_5 - DCFG	$\vec{k}$	yz	$dx\,dy$	$z=1$	y
S_6 - OABE	$-\vec{k}$	$-yz$	$dx\,dy$	$z=0$	0

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 4z\,dy\,dz = \int_0^1 (4yz)_0^1\,dz = \int_0^1 4z\,dz$$

$$= \left[\frac{4z^2}{2}\right]_0^1 = (2z^2)_0^1 = 2$$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = -\int_0^1\int_0^1 dx\,dy = -\int_0^1 (x)_0^1\,dy = -\int_0^1 dy = -[y]_0^1$$

$$= -1$$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 y\,dx\,dy = \int_0^1 [xy]_0^1\,dy = \int_0^1 y\,dy = \left[\frac{y^2}{2}\right]_0^1$$

$$= 1/2$$

$$\iint_{S} \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} \vec{F}\cdot\hat{n}\,ds + \iint_{S_3} \vec{F}\cdot\hat{n}\,ds + \iint_{S_5} \vec{F}\cdot\hat{n}\,ds$$

$$= 2 - 1 + 1/2$$

$$= 1 + 1/2 = 3/2.$$