Divergence of a vector point function.

Let $\vec{F}$ be any given continuously differentiable vector point function then the divergence of $\vec{F}$ is defined as,

$$\operatorname{div}\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\vec{F}$$

$$= \vec{i}\frac{\partial\vec{F}}{\partial x} + \vec{j}\frac{\partial\vec{F}}{\partial y} + \vec{k}\frac{\partial\vec{F}}{\partial z}$$

Note: 1. $\nabla.\vec{F}$ is a scalar point function

2. If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ be a continuously differentiable vector point function then,

$$\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Curl of a vector point function:

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ be any given continuously differentiable vector point function, the curl (or) rotation of $\vec{F}$ is defined as,

$$\operatorname{Curl}\vec{F} = \nabla\times\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\times\vec{F}$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\times\left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Note:- $\nabla\times\vec{F}$ is a vector point function.

Solenoidal vector:-

A vector $\vec{F}$ is said to be solenoidal vector if $\operatorname{div}\vec{F} = 0$

Irrotational vector:-

A vector $\vec{F}$ is said to be irrotational if $\nabla \times \vec{F} = 0$

ie $\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0$

Conservative vector Field:

If a vector point function $\vec{F}$ is expressible as the gradient of a scalar point function φ, then $\vec{F}$ is conservative ie) $\vec{F}$ is conservative if $\vec{F} = \nabla\varphi$. Here φ is called scalar potential. $\vec{F}$ is conservative force if $\text{curl}\,\vec{F} = 0$.

Problems:

1. Prove that $\text{curl}(\nabla\varphi) = 0$ (or) $\nabla \times \nabla\varphi = 0$

$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$

$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$

$\text{Curl}(\nabla\varphi) = \nabla \times \nabla\varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix}$

$= \vec{i}\left[\frac{\partial^2\varphi}{\partial y\partial z} - \frac{\partial^2\varphi}{\partial z\partial y}\right] - \vec{j}\left[\frac{\partial^2\varphi}{\partial x\partial z} - \frac{\partial^2\varphi}{\partial z\partial x}\right] + \vec{k}\left[\frac{\partial^2\varphi}{\partial x\partial y} - \frac{\partial^2\varphi}{\partial y\partial x}\right]$

$= 0.$

2. Prove that div (curl $\vec{F}$) = 0 (or) $\nabla \cdot (\nabla \times \vec{F}) = 0$.

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

$$\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] - \vec{j}\left[\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right] + \vec{k}\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$$

$$\nabla \cdot (\nabla \times \vec{F}) = \frac{\partial}{\partial x}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] - \frac{\partial}{\partial y}\left[\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right] + \frac{\partial}{\partial z}\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$$

$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial x \partial y} + \frac{\partial^2 F_1}{\partial y \partial z} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y}$$

$$= 0.$$

3. S.T curl grad $f = 0$ (or) $\nabla \times \nabla f = 0$.

$$\text{Curl grad } f = \nabla \times \nabla f = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right] - \vec{j}\left[\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right] + \vec{k}\left[\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right]$$

$$= 0.$$

4. If $\nabla v = y\vec{i} + z\vec{j} + x\vec{k}$. What is the directional derivative of v at the point $(1, 2, 3)$ in the direction $3\vec{i} + 4\vec{j} + 5\vec{k}$

$$\nabla v = y\vec{i} + z\vec{j} + x\vec{k} \qquad \nabla v_{(1,2,3)} = 2\vec{i} + 3\vec{j} + \vec{k}$$

$$\vec{a} = 3\vec{i} + 4\vec{j} + 5\vec{k} \Rightarrow |\vec{a}| = \sqrt{25+16+9} = \sqrt{50}$$

$$\text{Directional derivative} = \nabla v \cdot \frac{\vec{a}}{|\vec{a}|} = (2\vec{i} + 3\vec{j} + \vec{k}) \cdot \frac{(3\vec{i} + 4\vec{j} + 5\vec{k})}{\sqrt{50}}$$

$$= \frac{6+12+5}{\sqrt{50}}$$

$$= \frac{23}{\sqrt{50}}$$

5. Prove that $\operatorname{div}(\vec{u}\times\vec{v}) = \vec{v}\cdot \operatorname{curl}\vec{u} - \vec{u}\operatorname{curl}\vec{v}$

$$\operatorname{div}(\vec{u}\times\vec{v}) = \Sigma\vec{i}\,\frac{\partial}{\partial x}(\vec{u}\times\vec{v})$$

$$= \Sigma\vec{i}\left[\vec{u}\times\frac{\partial\vec{v}}{\partial x} + \frac{\partial\vec{u}}{\partial x}\times\vec{v}\right]$$

$$= \Sigma\vec{i}\left(\frac{\partial\vec{u}}{\partial x}\times\vec{v}\right) + \Sigma\vec{i}\left(\vec{u}\times\frac{\partial\vec{v}}{\partial x}\right)$$

$$= \left(\Sigma\vec{i}\times\frac{\partial\vec{u}}{\partial x}\right)\cdot\vec{v} - \left(\Sigma\vec{i}\times\frac{\partial\vec{v}}{\partial x}\right)\cdot\vec{u}$$

$$= \operatorname{curl}\vec{u}\cdot\vec{v} - \operatorname{curl}\vec{v}\cdot\vec{u}$$

$$= \vec{v}\operatorname{curl}\vec{u} - \vec{u}\cdot\operatorname{curl}\vec{v}.$$