VECTOR CALCULUS

Scalar quantity:

A scalar quantity is that which has magnitude and is not related to any direction

Vector quantity:

A vector quantity is that which has both magnitude and direction.

Scalar point function:

If corresponding to each point P of a region R there corresponds a scalar denoted by $\varphi(P)$ or $\varphi(x, y, z)$ the φ is said to be a scalar point function for the region R.

Example:

The temperature $\varphi(P)$ at any point P of a body occupying a certain region is a scalar point function.

Vector point function:

If corresponding to each point P of a region R, there corresponds a vector denoted by F(P) then F is said to be a vector point function for the Region R.

Example: The acceleration F(P) of a particle at any time t occupying the position P in a certain region is a vector point function.

Vector Differential operator:

The vector differential operator ∇ is defined as,

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} = \sum \vec{i}\frac{\partial}{\partial x}$$

Gradient of a scalar point function:

Let $\varphi(x, y, z)$ be a scalar point function and is continuously differentiable then the vector

$$\nabla\varphi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\varphi$$

$$= \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

is called the gradient of φ and is written as grad φ ie, $\text{grad}\,\varphi = \nabla\varphi$

Note:

1. $\nabla\varphi$ defines a vector field
2. $\nabla\varphi \neq \varphi\nabla$. There will be no '.' or 'x' between φ and ∇.

Properties of Gradient:

1. If f and g are two scalar point functions then,

$$\nabla(f \pm g) = \nabla f \pm \nabla g$$

or $\text{grad}(f \pm g) = \text{grad}\, f \pm \text{grad}\, g$

2. If f and g are two scalar point functions then

$$\nabla(fg) = f\nabla g + g\nabla f$$

(or) $$\text{grad}(fg) = f(\text{grad } g) + g(\text{grad } f)$$

3. If f and g are two scalar point functions then,

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2} \quad \text{where } g \neq 0$$

(or) $$\text{grad}\left(\frac{f}{g}\right) = \frac{g(\text{grad } f) - f(\text{grad } g)}{g^2}$$

4. Gradient of a constant is zero.

$$\nabla\varphi = 0.$$

Problems:

1. Find $\text{grad}\,\varphi$ where $\varphi = x^2 + y^2 + z^2$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2+y^2+z^2) + \vec{j}\frac{\partial}{\partial y}(x^2+y^2+z^2) + \vec{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$= \vec{i}(2x) + \vec{j}(2y) + \vec{k}(2z)$$

$$\nabla\varphi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

2. Find $\text{grad}\,\varphi$ if $\varphi = xyz$ at $(1,1,1)$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\nabla\varphi = \vec{i}\frac{\partial}{\partial x}(xyz) + \vec{j}\frac{\partial}{\partial y}(xyz) + \vec{k}\frac{\partial}{\partial z}(xyz)$$

$$= \vec{i}(yz) + \vec{j}(xz) + \vec{k}(xy)$$

$$\nabla\varphi_{(1,1,1)} = \vec{i} + \vec{j} + \vec{k}$$

3. Find grad φ where $\varphi = 3x^2y - y^3z^2$ at $(1,1,1)$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \vec{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \vec{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= \vec{i}(6xy) + \vec{j}(3x^2 - 3y^2z^2) + \vec{k}(-2y^3z)$$

$$\nabla\varphi_{(1,1,1)} = 6\vec{i} + \vec{k}(-2)$$

$$= 6\vec{i} - 2\vec{k}$$

4. If $\varphi = \log(x^2+y^2+z^2)$ find $\nabla\varphi$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}[\log(x^2+y^2+z^2)] + \vec{j}\frac{\partial\varphi}{\partial y}(\log(x^2+y^2+z^2)) + \vec{k}\frac{\partial}{\partial z}(\log(x^2+y^2+z^2))$$

$$= \vec{i}\frac{1}{x^2+y^2+z^2}(2x) + \vec{j}\frac{1}{x^2+y^2+z^2}(2y) + \vec{k}\frac{1}{x^2+y^2+z^2}(2z)$$

$$= \frac{2}{x^2+y^2+z^2}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\nabla\varphi = \frac{2\vec{r}}{x^2+y^2+z^2}$$

5) Find $\nabla(\log r)$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\nabla(\log r) = \vec{i}\frac{\partial}{\partial x}(\log r) + \vec{j}\frac{\partial}{\partial y}(\log r) + \vec{k}\frac{\partial}{\partial z}(\log r)$$

$$= \vec{i}\frac{1}{r}\frac{\partial r}{\partial x} + \vec{j}\frac{1}{r}\frac{\partial r}{\partial y} + \vec{k}\frac{1}{r}\frac{\partial r}{\partial z} \rightarrow (1)$$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

since $r = \sqrt{x^2+y^2+z^2}$

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ & $\frac{\partial r}{\partial z} = \frac{z}{r}$

sub the values in (1)

$$\nabla(\log r) = \vec{i}\frac{1}{r}\left(\frac{x}{r}\right) + \vec{j}\frac{1}{r}\left(\frac{y}{r}\right) + \vec{k}\cdot\frac{1}{r}\cdot\left(\frac{z}{r}\right)$$

$$= \frac{1}{r^2}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\nabla(\log r) = \frac{\vec{r}}{r^2}$$

(6) If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ such that $|\vec{r}| = r$,

Prove that

i) $\nabla r = \frac{\vec{r}}{r} = \hat{r}$

ii) $\nabla\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3} = \frac{\hat{r}}{r^2}$

iii) $\nabla r^n = n r^{n-2}\vec{r}$

iv) $\nabla f(r) = f'(r)\nabla r$

v) $\nabla f(r) \times \vec{r} = 0$

vi) If $\nabla\varphi$ is solenoidal find $\nabla^2\varphi$

Solution:

i) Given: $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

$$2r \cdot \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$2r \frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$2r \frac{\partial r}{\partial z} = 2z = \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla r = i\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}$$

$$= \vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}$$

$$= \frac{1}{r}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\boxed{\nabla r = \frac{\vec{r}}{r} = \hat{r}}$$

ii) $\nabla\left(\frac{1}{r}\right) = \vec{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \vec{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \vec{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$

$$= \vec{i}\left(\frac{-1}{r^2}\frac{\partial r}{\partial x}\right) + \vec{j}\left(\frac{-1}{r^2}\frac{\partial r}{\partial y}\right) + \vec{k}\left(\frac{-1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= \frac{-1}{r^2}\left[\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right]$$

$$\nabla\left(\frac{1}{r}\right) = \frac{-1}{r^3}(\vec{r}) = \frac{-1}{r^2}\left(\frac{\vec{r}}{r}\right) = \frac{-\hat{r}}{r^2}$$

$$\nabla\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3} = \frac{-\hat{r}}{r^2}$$

iii) $\nabla r^n = \left(\vec{i} \frac{\partial r^n}{\partial x} + \vec{j} \frac{\partial r^n}{\partial y} + \vec{k} \frac{\partial r^n}{\partial z} \right)$

$= \vec{i}\, n r^{n-1} \frac{\partial r}{\partial x} + \vec{j}\, n r^{n-1} \frac{\partial r}{\partial y} + \vec{k}\, n r^{n-1} \frac{\partial r}{\partial z}$

$= n r^{n-1} \left[\vec{i} \frac{x}{r} + \vec{j} \frac{y}{r} + \vec{k} \frac{z}{r} \right]$

$= \frac{n r^{n-1}}{r} [x\vec{i} + y\vec{j} + z\vec{k}]$

$\nabla r^n = n r^{n-2} \vec{r}$

iv) $\nabla f(r) = \vec{i} \frac{\partial}{\partial x} f(r) + \vec{j} \frac{\partial}{\partial y} f(r) + \vec{k} \frac{\partial}{\partial z} f(r)$

$= \vec{i} f'(r) \frac{\partial r}{\partial x} + \vec{j} f'(r) \frac{\partial r}{\partial y} + \vec{k} f'(r) \frac{\partial r}{\partial z}$

$= f'(r) \left[\vec{i} \frac{x}{r} + \vec{j} \frac{y}{r} + \vec{k} \frac{z}{r} \right]$

$= \frac{f'(r)}{r} (x\vec{i} + y\vec{j} + z\vec{k})$

$\nabla f(r) = \frac{f'(r)}{r} \vec{r}$

v) $\nabla f(r) \times \vec{r} = \frac{f'(r) \vec{r}}{r} \times \vec{r}$

$= \frac{1}{r} f'(r) [\vec{r} \times \vec{r}] = 0.$ $\quad \because \vec{r} \times \vec{r} = 0$

$\therefore \nabla f(r) \times \vec{r} = 0.$

vi) $\nabla^2 \varphi = \nabla (\nabla \varphi)$ $\quad (\because \nabla\varphi$ is solenoidal $\nabla\varphi = 0)$

$= \nabla(0)$

$\nabla^2 \varphi = 0.$

Level Surface : Important Results

Unit Normal

A unit normal to the given surface φ at the point is $\frac{\nabla\varphi}{|\nabla\varphi|}$

Directional Derivative :-

The directional derivative of φ in the direction $\vec{a}$ is given by

$$\nabla\varphi \cdot \frac{\vec{a}}{|\vec{a}|} \quad \text{(or)} \quad \nabla\varphi \cdot \hat{n} \quad \text{where} \quad \hat{n} = \frac{\vec{a}}{|\vec{a}|}$$

The directional derivative is maximum in the direction of the normal to the given surface.

Its maximum value is $|\nabla\varphi|$

Angle between two Surfaces:

$$\cos\theta = \frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1||\nabla\varphi_2|}$$

Note:

If the surfaces cut orthogonally then,

$$\nabla\varphi_1 \cdot \nabla\varphi_2 = 0.$$

Problems:

1. Find a unit normal to the surface $x^2y + 2xz = 4$ at $(2, -2, 3)$

soln: $\varphi = x^2y + 2xz - 4$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2y + 2zx - 4) + \vec{j}\frac{\partial}{\partial y}(x^2y + 2xz - 4) + \vec{k}\frac{\partial}{\partial z}(x^2y + 2xz - 4)$$

$$= \vec{i}(2xy + 2z) + \vec{j}(x^2) + \vec{k}(2x)$$

$$\nabla\varphi_{(2,-2,3)} = \vec{i}(-8+6) + \vec{j}(4) + \vec{k}(4)$$

$$= -2\vec{i} + 4\vec{j} + 4\vec{k}$$

$$|\nabla\varphi| = \sqrt{4+16+16} = \sqrt{36} = 6$$

Unit normal to the given surface at $(2, -2, 3)$

$$= \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{-2\vec{i} + 4\vec{j} + 4\vec{k}}{6}$$

$$= \frac{1}{3}(-\vec{i} + 2\vec{j} + 2\vec{k})$$

2. Find the unit normal to $x^2 - y^2 + z = 2$ at $(1, -1, 2)$

solution:-

$$\varphi = x^2 - y^2 + z - 2$$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2 - y^2 + z - 2) + \vec{j}\frac{\partial}{\partial y}(x^2 - y^2 + z - 2) + \vec{k}\frac{\partial}{\partial z}(x^2 - y^2 + z - 2)$$

$$= \vec{i}(2x) + \vec{j}(-2y) + \vec{k}(1)$$

$$= 2x\vec{i} + \vec{k} - 2y\vec{j}$$

$$\nabla\varphi_{(1,-1,2)} = \vec{i}(2(1)) - \vec{j}(2(-1)) + \vec{k}(1)$$

$$= 2\vec{i} + 2\vec{j} + \vec{k}$$

$$|\nabla\varphi| = \sqrt{2^2 + (+2)^2 + 1^2} = \sqrt{4+4+1}$$

$$= \sqrt{9}$$

$$= 3.$$

$$\frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2\vec{i} + 2\vec{j} + \vec{k}}{3}$$

3. Find the unit vector normal to $x^2 + xy + z^2 = 4$ at $(1,-1,2)$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= i(2x+y) + \vec{j}(x) + \vec{k}(2z)$$

$$\nabla\varphi_{(1,-1,2)} = \vec{i}(2(1) + (-1)) + \vec{j}(1) + \vec{k}(2(2))$$

$$= \vec{i} + \vec{j} + 4\vec{k}$$

$$|\nabla\varphi| = \sqrt{1+1+16} = \sqrt{18}$$

$$\frac{\nabla\varphi}{|\nabla\varphi|} = \frac{\vec{i} + \vec{j} + 4\vec{k}}{\sqrt{18}}$$

4. Find the directional derivative of the function x^2+2xy at $(1,-1,3)$ in the direction $\vec{i}+2\vec{j}+2\vec{k}$

Soln: $\varphi = x^2+2xy$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2+2xy) + \vec{j}\frac{\partial}{\partial y}(x^2+2xy) + \vec{k}\frac{\partial}{\partial z}(x^2+2xy)$$

$$= \vec{i}\,(2x+2y) + \vec{j}\,(2x) + \vec{k}(0)$$

$$\nabla\varphi_{(1,-1,3)} = \vec{i}(2\times1+2(-1)) + \vec{j}\,(2(1))$$

$$= 2\vec{j}$$

Given:- $\vec{a} = \vec{i}+2\vec{j}+2\vec{k}$

$$|\vec{a}| = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$\hat{n} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{i}+2\vec{j}+2\vec{k}}{3}$$

$$\nabla\varphi\cdot\hat{n} = 2\vec{j}\cdot\left[\frac{\vec{i}+2\vec{j}+2\vec{k}}{3}\right] = \frac{4}{3}$$

$$\therefore\ \nabla\varphi\cdot\hat{n} = \frac{4}{3}$$

~~5.~~ Find the directional derivative of $xy+yz+zx$ at $(1,1,1)$ in the direction $\vec{i}+\vec{j}$
$(1,2,0)$ $\quad \vec{i}+2\vec{j}+2\vec{k}$

$\varphi = xy+yz+zx$

$$\nabla\varphi = \vec{i}\frac{\partial}{\partial x}(xy+yz+zx) + \vec{j}\frac{\partial}{\partial y}(xy+yz+zx) + \vec{k}\frac{\partial}{\partial z}(xy+yz+zx)$$

$$= \vec{i}\,(y+z) + \vec{j}\,(x+z) + \vec{k}\,(y+x)$$

$$\nabla\varphi_{(1,1,1)} = 2\vec{i}+2\vec{j}+2\vec{k}$$

$$= 2(\vec{i}+\vec{j}+\vec{k})$$

Given: $\vec{a} = \vec{i} + \vec{j}$

$|\vec{a}| = \sqrt{1^2+1^2} = \sqrt{2}$

$\hat{n} = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{\vec{i}+\vec{j}}{\sqrt{2}}$

Directional derivative $= \nabla\varphi \cdot \hat{n}$

$= 2(\vec{i}+\vec{j}+\vec{k}) \cdot \dfrac{\vec{i}+\vec{j}}{\sqrt{2}}$

$= \dfrac{2}{\sqrt{2}}(1+1)$

$= 2\sqrt{2}$

(b) Find the directional derivative of $3x^2+2y-3z$ at $(1,1,1)$ in the direction $2\vec{i}+2\vec{j}-\vec{k}$

$\varphi = 3x^2+2y-3z$

$\nabla\varphi = \vec{i}\dfrac{\partial}{\partial x}(3x^2+2y-3z) + \vec{j}\dfrac{\partial}{\partial y}(3x^2+2y-3z) + \vec{k}\dfrac{\partial}{\partial z}(3x^2+2y-3z)$

$= \vec{i}(6x) + \vec{j}(2) + \vec{k}(-3)$

$= 6x\vec{i} + 2\vec{j} - 3\vec{k}$

$\nabla\varphi_{(1,1,1)} = 6\vec{i} + 2\vec{j} - 3\vec{k}$

Given: $\vec{a} = 2\vec{i}+2\vec{j}-\vec{k}$ $\quad |\vec{a}| = \sqrt{2^2+2^2+(-1)^2} = \sqrt{9} = 3$

$\dfrac{\vec{a}}{|\vec{a}|} = \dfrac{2\vec{i}+2\vec{j}-\vec{k}}{3}$

$\nabla\varphi \cdot \hat{n} = \dfrac{2\vec{i}+2\vec{j}-\vec{k}}{3} \cdot 6\vec{i}+2\vec{j}-3\vec{k}$

$= \dfrac{12+4+3}{3} = \dfrac{19}{3}$

7. What is the greatest rate of increase of $\varphi = xyz^2$ at $(1,0,3)$?

Let $\varphi = xyz^2$

$$\nabla\varphi = \vec{i}\frac{\partial}{\partial x}(xyz^2) + \vec{j}\frac{\partial}{\partial y}(xyz^2) + \vec{k}\frac{\partial}{\partial z}(xyz^2)$$

$$= \vec{i}(yz^2) + \vec{j}(xz^2) + \vec{k}(2xyz)$$

$$\nabla\varphi_{(1,0,3)} = 9\vec{j}$$

Maximum (or) Greatest rate of increase $= |\nabla\varphi| = \sqrt{9^2}$

$= 9$

8. In what direction from the point $(1,-1,2)$ is the directional derivative of $\varphi = x^2y^2z^2$ a maximum? What is the magnitude of this maximum? x^2yz^3 at $(1,2,-1)$

$\varphi = x^2y^2z^2$

$$\nabla\varphi = \vec{i}\frac{\partial}{\partial x}(x^2y^2z^2) + \vec{j}\frac{\partial}{\partial y}(x^2y^2z^2) + \vec{k}\frac{\partial}{\partial z}(x^2y^2z^2)$$

$$= 2xy^2z^2\vec{i} + 2yx^2z^2\vec{j} + 2x^2y^2z\vec{k}$$

$$\nabla\varphi_{(1,-1,2)} = 2(1)(-1)^2(2)^2\vec{i} + 2(-1)(1)^2(2)^2\vec{j} + 2(1)^2(-1)^2(2)\vec{k}$$

$= 8\vec{i} - 8\vec{j} + 4\vec{k}$ is the directional derivative

Magnitude is $|\nabla\varphi| = \sqrt{8^2 + (-8)^2 + 4^2}$

$= \sqrt{64+64+16}$

$= 12.$

Q Find the directional derivative of $\varphi = xy^2z^3$ at the point $(1,1,1)$ along the normal to the surface $x^2+xy+z^2=3$ at the point $(1,1,1)$

$(-1,1,2)$

$\nabla\varphi$ is normal to the surface x^2+xy+z^2-3

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2+xy+z^2-3) + \vec{j}\frac{\partial}{\partial y}(x^2+xy+z^2-3) + \vec{k}\frac{\partial}{\partial z}(x^2+xy+z^2-3)$$

$$= \vec{i}(2x+y) + \vec{j}(x) + \vec{k}(2z)$$

$$\nabla\varphi_{(1,1,1)} = 3\vec{i}+\vec{j}+2\vec{k}$$

To find the directional derivative of $\varphi = xy^2z^3$ at $(1,1,1)$ in the direction $\vec{a} = 3\vec{i}+\vec{j}+2\vec{k}$

$$\nabla\varphi = \vec{i}\frac{\partial}{\partial x}(xy^2z^3) + \vec{j}\frac{\partial}{\partial y}(xy^2z^3) + \vec{k}\frac{\partial}{\partial z}(xy^2z^3)$$

$$= \vec{i}(y^2z^3) + \vec{j}(2xyz^3) + \vec{k}(3xy^2z^2)$$

$$\nabla\varphi_{(1,1,1)} = \vec{i} + 2\vec{j} + 3\vec{k}$$

$$\text{Directional derivative} = \nabla\varphi \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= \vec{i}+2\vec{j}+3\vec{k} \cdot \frac{(3\vec{i}+\vec{j}+2\vec{k})}{\sqrt{3^2+1^2+2^2}}$$

$$= \frac{3+2+6}{\sqrt{14}} = \frac{11}{\sqrt{14}}$$

10. Find the angle between the surfaces $x^2+y^2+z^2=5$ and $x^2+y^2+z^2-2x=5$ at $(0,1,2)$

Let $\varphi_1 = x^2+y^2+z^2-5$; $\varphi_2 = x^2+y^2+z^2-2x-5$

ϕ φ

$\frac{\partial\varphi_1}{\partial x} = 2x$ $\qquad$ $\frac{\partial\varphi_2}{\partial x} = 2x-2$

$\frac{\partial\varphi_2}{\partial y} = 2y$ $\qquad$ $\frac{\partial\varphi_2}{\partial y} = 2y$

$\frac{\partial\varphi_2}{\partial z} = 2z$ $\qquad$ $\frac{\partial\varphi_2}{\partial z} = 2z$

$\nabla\varphi_1 = 2x\vec{i}+2y\vec{j}+2z\vec{k}$; $\nabla\varphi_2 = (2x-2)\vec{i}+2y\vec{j}+2z\vec{k}$

$\nabla\varphi_1(0,1,2) = 2\vec{j}+4\vec{k}$ $\sqrt{2^2+4^2}$ $\quad$ $\nabla\varphi_2(0,1,2) = -2\vec{i}+2\vec{j}+4\vec{k}$

$|\nabla\varphi_1| = \sqrt{4+16} = \sqrt{20}$ $\qquad$ $|\nabla\varphi_2| = \sqrt{4+4+16}$
$= \sqrt{24}$

Angle between the surfaces, $\quad \cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

$$\cos\theta = \frac{\nabla\varphi_1\cdot\nabla\varphi_2}{|\nabla\varphi_1||\nabla\varphi_2|}$$

$$= \frac{(2\vec{j}+4\vec{k})\cdot(-2\vec{i}+2\vec{j}+4\vec{k})}{\sqrt{20}\cdot\sqrt{24}}$$

$$= \frac{4+16}{\sqrt{20}.\sqrt{24}} = \frac{20}{\sqrt{20}.\sqrt{24}} = \sqrt{\frac{20}{24}}$$

$$\cos\theta = \sqrt{\frac{5}{6}}$$

$$\theta = \cos^{-1}\sqrt{\frac{5}{6}}$$

11. Find the angle between the surfaces $x \log z = y^2 - 1$ and $x^2 y = 2 - z$ at the point $(1,1,1)$

$x = y^2 - 1$

$x^2 y = 2$

$\varphi_1 = x \log z - y^2 + 1 \quad ; \quad \varphi_2 = x^2 y - 2 + z$ at $(1,1,1)$

$\frac{\partial \varphi_1}{\partial x} = \log z \qquad \frac{\partial \varphi_2}{\partial x} = 2xy$

$\frac{\partial \varphi_1}{\partial y} = -2y \qquad \frac{\partial \varphi_2}{\partial y} = x^2$

$\frac{\partial \varphi_1}{\partial z} = \frac{x}{z} \qquad \frac{\partial \varphi_2}{\partial z} = 1$

$\nabla \varphi_1 = \log z \, \vec{i} - 2y \, \vec{j} + \frac{x}{z} \vec{k} \quad ; \quad \nabla \varphi_2 = 2xy \, \vec{i} + x^2 \vec{j} + \vec{k}$

$\nabla \varphi_{1(1,1,1)} = -2\vec{j} + \vec{k} \qquad \nabla \varphi_{2(1,1,1)} = 2\vec{i} + \vec{j} + \vec{k}$

$|\nabla \varphi_1| = \sqrt{4+1} = \sqrt{5} \qquad |\nabla \varphi_2| = \sqrt{4+1+1} = \sqrt{6}$

$$\cos\theta = \frac{\nabla \varphi_1 \cdot \nabla \varphi_2}{|\nabla \varphi_1| \, |\nabla \varphi_2|}$$

$$= \frac{(-2\vec{j} + \vec{k}) \cdot (2\vec{i} + \vec{j} + \vec{k})}{\sqrt{5}\sqrt{6}}$$

$$= \frac{-2+1}{\sqrt{5}\sqrt{6}}$$

$$= \frac{-1}{\sqrt{5}\sqrt{6}}$$

$$\cos\theta = \frac{-1}{\sqrt{30}} \qquad \theta = \cos^{-1}\left(\frac{-1}{\sqrt{30}}\right)$$

12. Find the angle between the surfaces $x^2+y^2+z^2=9$

$z = x^2+y^2-2$ at $(2,-1,2)$

$\varphi_1 = x^2+y^2+z^2-9$; $\varphi_2 = x^2+y^2-2-z$

$\nabla\varphi_1 \Rightarrow$

$\frac{\partial\varphi_1}{\partial x} = 2x$ $\qquad$ $\frac{\partial\varphi_2}{\partial x} = 2x$

$\frac{\partial\varphi_1}{\partial y} = 2y$ $\qquad$ $\frac{\partial\varphi_2}{\partial y} = 2y$

$\frac{\partial\varphi_1}{\partial z} = 2z$ $\qquad$ $\frac{\partial\varphi_2}{\partial z} = -1$

$\nabla\varphi_1 = 2x\vec{i}+2y\vec{j}+2z\vec{k}$ $\qquad$ $\nabla\varphi_2 = 2x\vec{i}+2y\vec{j}-\vec{k}$

$\nabla\varphi_1(2,-1,2) = 4\vec{i}-2\vec{j}+4\vec{k}$ $\qquad$ $\nabla\varphi_2(2,-1,2) = 4\vec{i}-2\vec{j}-\vec{k}$

$|\nabla\varphi_1| = \sqrt{16+4+16} = \sqrt{36} = 6$; $|\nabla\varphi_2| = \sqrt{16+4+1} = \sqrt{21}$

$$\cos\theta = \frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1|\,|\nabla\varphi_2|}$$

$$= \frac{(4\vec{i}-2\vec{j}+4\vec{k}).(4\vec{i}-2\vec{j}-\vec{k})}{(6)(\sqrt{21})}$$

$$= \frac{16+4-4}{6\sqrt{21}} = \frac{16}{6\sqrt{21}}$$

$$\cos\theta = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

13. Find a and b such that surfaces $ax^2+byz=(a+2)z$ and $4x^2y+z^3=4$ cuts orthogonally at $(1,-1,2)$.

Let $\varphi_1 = ax^2+byz-(a+2)x$

$\varphi_2 = 4x^2y+z^3-4$

$$\nabla\varphi_1 = \vec{i}\frac{\partial}{\partial x}(ax^2+byz-(a+2)x)+\vec{j}\frac{\partial}{\partial y}(ax^2+byz-(a+2)x)+\vec{k}\frac{\partial}{\partial z}(ax^2+byz-(a+2)x)$$

$$= \vec{i}\,(2ax-a-2)+\vec{j}\,(bz)+\vec{k}\,(by)$$

$$\nabla\varphi_{1\,(1,-1,2)} = (a-2)\vec{i}+2b\vec{j}-b\vec{k}$$

$$\nabla\varphi_2 = \vec{i}\frac{\partial}{\partial x}(4x^2y+z^3-4)+\vec{j}\frac{\partial}{\partial y}(4x^2y+z^3-4)+\vec{k}\frac{\partial}{\partial z}(4x^2y+z^3-4)$$

$$= \vec{i}(8xy)+\vec{j}(4x^2)+\vec{k}(3z^2)$$

$$\nabla\varphi_{2\,(1,-1,2)} = -8\vec{i}+4\vec{j}+12\vec{k}$$

Since the surface cuts orthogonally,

$$\nabla\varphi_1\cdot\nabla\varphi_2 = 0$$

$$[(a-2)\vec{i}+2b\vec{j}-b\vec{k}]\cdot[-8\vec{i}+4\vec{j}+12\vec{k}]=0$$

$$-8a+16+8b-12b=0$$

$$\cancel{-8b}\ -8a-4b=-16$$

$$2a+b=4 \rightarrow (1)$$

Since the point $(1,-1,2)$ lies on φ

$$a-2b-(a+2)=0 \Rightarrow \boxed{b=-1}$$

Sub $b=-1$ in (1) $\rightarrow \boxed{a=5/2}$

14. Find the values of a and b so that the surface $ax^3 - by^2z = (a+3)x^2$ and $4x^2y - z^3 = 11$ may cut orthogonally at $(2,-1,-3)$

Let $\varphi_1 = ax^3 - by^2z - (a+3)x^2$; $\varphi_2 = 4x^2y - z^3 - 11$

$$\nabla\varphi_1 = \vec{i}\,(3ax^2 - 2(a+3)x) + \vec{j}\,(-6b) + \vec{k}\,(-b)$$

$$\nabla\varphi_1(2,-1,-3) = \vec{i}\,(12a - 4a - 12) + \vec{j}\,(-6b) + \vec{k}\,(-b)$$

$$= \vec{i}\,(8a-12) + \vec{j}\,(-6b) + \vec{k}\,(-b)$$

$$\nabla\varphi_2 = \vec{i}\,(8xy) + \vec{j}\,(4x^2) + \vec{k}\,(-3z^2)$$

$$\nabla\varphi_2(2,-1,-3) = -16\vec{i} + 16\vec{j} - 27\vec{k}$$

$\because$ The surface cuts orthogonally

$$\nabla\varphi_1 \cdot \nabla\varphi_2 = 0$$

$$[(8a-12)\vec{i} - 6b\vec{j} - b\vec{k}] \cdot [-16\vec{i} + 16\vec{j} - 27\vec{k}] = 0$$

$$(8a-12)(-16) - (6b)(16) + 27b = 0$$

$$-128a + 192 - 96b + 27b = 0.$$

$$-128a - 69b + 192 = 0$$

$$128a + 69b = 192 \rightarrow (1)$$

$\because$ The point $(2,-1,-3)$ lie on φ,

$$8a + 3b - 4(a+3) = 0$$

$$8a + 3b - 4a - 12 = 0$$

$$\Rightarrow 4a + 3b = 12 \Rightarrow (2)$$

⊗ by 23 in ②

$\Rightarrow \quad 128a + 69b = 192$

$(-)\ 92a + 69b = 276$

$36a = -84$

$a = \frac{-84}{36} = \frac{-7}{3}$

$\therefore a = \frac{-7}{3}$

Sub a in ② $\Rightarrow 4\left(\frac{-7}{3}\right) + 3b = 12$

$-\frac{28}{3} + 3b = 12$

$3b = 12 + \frac{28}{3}$

$= \frac{36+28}{3}$

$b = \frac{64}{9}$

$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\ldots$

find the angle b/w the normals to the surface $xy = z^2$ at the pt (1, 4, 2), (−3, −3, 3)

$\phi = xy - z^2$

$\nabla\phi = y\vec{i} + x\vec{j} - 2z\vec{k}$

$\nabla\phi_{1(1,4,2)} = 4\vec{i} + \vec{j} - 4\vec{k}$

$|\nabla\phi_1| = \sqrt{33}$

$\nabla\phi_2 = -3\vec{i} - 3\vec{j} - 6\vec{k}$

$|\nabla\phi_2| = \sqrt{54}$

$\cos\theta = \frac{9}{\sqrt{33}\sqrt{54}} = \frac{1}{\sqrt{22}}$

$= \frac{-12 - 3 + 24}{\sqrt{33}\sqrt{54}} = \frac{9}{\sqrt{3 \times 11 \times 2 \times 3 \times 3 \times 3}}$

$\theta = \cos^{-1}\left(1/\sqrt{22}\right)$