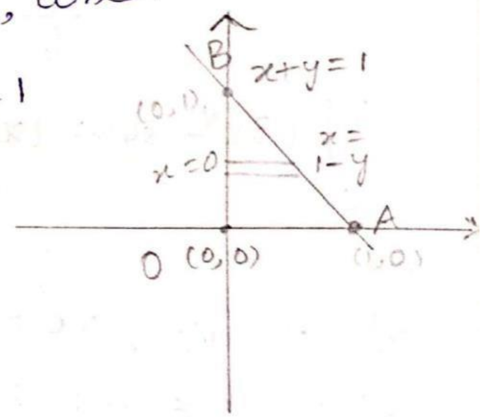
4]. Use green's theorem & evaluate $\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$, where C is the curve with $x=0,\ y=0,\ x+y=1$



Soln.

Given $x=0,\ y=0,\ x+y=1$

when $x=0,\ y=1 \Rightarrow (0,1)$

$y=0,\ x=1 \Rightarrow (1,0)$

By Green's theorem,

$$\int_C [M\,dx + N\,dy] = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$

Here $M = 3x^2 - 8y^2$ | $N = 4y - 6xy$

$\frac{\partial M}{\partial y} = -16y$ | $\frac{\partial N}{\partial x} = -6y$

RHS

$$\iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy = \int_0^1 \int_0^{1-y} [-6y + 16y]\,dx\,dy$$

$$= \int_0^1 \int_0^{1-y} 10\, y \, dx \, dy$$

$$= 10 \int_0^1 \int_0^{1-y} y \, dx \, dy = 10 \int_0^1 y \, [x]_0^{1-y} \, dy$$

$$= 10 \int_0^1 y \, [1 - y - 0] \, dy$$

$$= 10 \int_0^1 [y - y^2] \, dy$$

$$= 10 \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_0^1 = 10 \left[\frac{1}{2} - \frac{1}{3} \right]$$

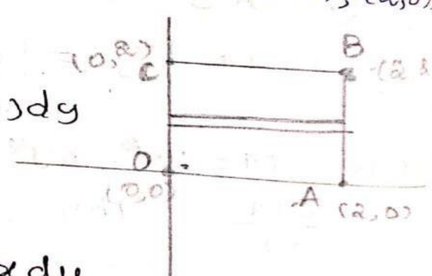
$$= 10 \left[\frac{3-2}{6} \right]$$

$$= \frac{10}{6}$$

$$\therefore \int_C (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy = \frac{5}{3}$$

5] verify Green's theorem $\int_C (x^2 - xy^3)dx + (y^2 - 2xy)dy$ where c is the square with vertices (0, 0), (2,0), (2, 2), (0, 2).

Soln.



Given $\int_C (x^2 - xy^3)\, dx + (y^2 - 2xy)\, dy$

To verify

$$\int_C M\,dx + N\,dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] dx\, dy$$

Here $M = x^2 - xy^3$ | $N = y^2 - 2xy$

$\frac{\partial M}{\partial y} = -3xy^2$ | $\frac{\partial N}{\partial x} = -2y$

RHS

$$\iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy = \int_0^2\int_0^2 [-2y + 3xy^2]\,dx\,dy$$

$$= \int_0^2 \left[-2yx + 3\frac{x^2}{2}y^2\right]_{x=0}^{2} dy$$

$$= \int_0^2 \left[-4y + \frac{3}{2}(4)y^2\right] dy$$

$$= \int_0^2 [-4y + 6y^2]\,dy$$

$$= \left[-\frac{4y^2}{2} + \frac{6y^3}{3}\right]_0^2$$

$$= [-2(4) + 2(8)] - 0$$

$$= 8$$

LHS

To evaluate $\int_C [M\,dx + N\,dy]$, we shall take C in the different paths.

i). Along OA $[y=0]$

ii). Along AB $[x=2]$

iii). Along BC $[y=2]$

iv). Along CO $[x=0]$

Along OA $[y=0 \Rightarrow dy=0]$

$$\int_{OA} (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$$

$$= \int_0^2 [x^2 - 0]\,dx + [0-0](0)$$

$$= \int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2$$

Along AB ($x = 2 \Rightarrow dx = 0$)

$$\int_{AB} (x^2 - xy^3)dx + (y^2 - 2xy)dy$$

$$= \int_0^2 (4 - 2y^3)(0) + (y^2 - 4y)dy$$

$$= \int_0^2 [y^2 - 4y]\, dy$$

$$= \left[\frac{y^3}{3} - \frac{4y^2}{2}\right]_0^2$$

$$= \left(\frac{8}{3} - 2(4)\right) - 0 = \frac{8}{3} - 8$$

$$= \frac{8 - 24}{3}$$

$$= \frac{-16}{3}$$

Along BC ($y = 2 \Rightarrow dy = 0$)

$$\int_{BC} (x^2 - xy^3)dx + (y^2 - 2xy)\, dy$$

$$= \int_2^0 (x^2 - x(8))dx + 0$$

$$= \int_2^0 [x^2 - 8x]\, dx$$

$$= \left[\frac{x^3}{3} - 8\frac{x^2}{2}\right]_2^0$$

$$= 0 - \left(\frac{8}{3} - 4(4)\right) = -\left[\frac{8 - 48}{3}\right]$$

$$= \frac{40}{3}$$

Along CO

$x=0 \Rightarrow dx=0$

$$\int_{CO} (x^2-xy^3)\,dx + (y^2-2xy)\,dy = \int_2^0 [0 + (y^2-0)\,dy]$$

$$= \int_2^0 y^2\,dy$$

$$= \left[\frac{y^3}{3}\right]_2^0$$

$$= 0 - 8/3$$

$$= -8/3$$

$$\therefore \int_C (x^2-xy^3)\,dx + (y^2-2xy)\,dy = \frac{-8 \cdot 16+40+8}{3}$$

$$= \frac{24}{3}$$

$$= 8$$

$\therefore$ LHS = RHS

Hence verified.