Divergence and curl:

1. Div $\vec{F} = \nabla \cdot \vec{F}$
2. curl $\vec{F} = \nabla \times \vec{F}$

Problems:

1. If $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$, then find $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$

Soln.

Given $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 2x + 2y + 2z$$

$$\nabla \cdot \vec{F} = 2(x+y+z)$$

and
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(y^2)\right] - \vec{j}\left[\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(x^2)\right] + \vec{k}\left[\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2)\right]$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$$\nabla \times \vec{F} = \vec{0}$$

2. Determine the constant 'a' so that the vector $\vec{F} = (x+z)\vec{i} + (3x+ay)\vec{j} + (x-5z)\vec{k}$ is such that its divergence is zero.

Soln.

Given $\nabla \cdot \vec{F} = 0$

Now
$$\frac{\partial}{\partial x}(x+z) + \frac{\partial}{\partial y}(3x+ay) + \frac{\partial}{\partial z}(x-5z) = 0$$

$$1 + a - 5 = 0$$

$$\boxed{a = 4}$$

3]. Find $\nabla \cdot \left(\frac{1}{r}\vec{r}\right)$

Soln.

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\frac{1}{r}\vec{r} = \frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k}$$

Now, $\nabla \cdot \left(\frac{1}{r}\vec{r}\right) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(\frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$= \frac{r(1) - x\frac{\partial r}{\partial x}}{r^2} + \frac{r(1) - y\frac{\partial r}{\partial y}}{r^2} + \frac{r(1) - z\frac{\partial r}{\partial z}}{r^2}$$

$$= \frac{1}{r^2}\left[r - x\left(\frac{x}{r}\right) + r - y\left(\frac{y}{r}\right) + r - z\left(\frac{z}{r}\right)\right]$$

$$= \frac{1}{r^2}\left[3r - \frac{x^2}{r} - \frac{y^2}{r} - \frac{z^2}{r}\right]$$

$$= \frac{1}{r^2}\left[3r - \frac{1}{r}(x^2 + y^2 + z^2)\right]$$

$$= \frac{1}{r^2}\left[3r - \frac{1}{r}r^2\right] = \frac{1}{r^2}\left[3r - r\right]$$

$$= \frac{1}{r^2}(2r)$$

$$= \frac{2}{r}$$

$$\nabla \cdot \left(\frac{1}{r}\vec{r}\right) = \frac{2}{r}$$