

Cross correlation :

A cross correlation of a two Random Processes $x(t)$ & $y(t)$ is defined by,

$$R_{xy}(\tau) = E[x(t) y(t+\tau)]$$

$$R_{yx}(\tau) = E[y(t) \cdot x(t+\tau)]$$

1. If $x(t)$ and $y(t)$ are WSS process, then

$$R_{xy}(-\tau) = R_{yx}(\tau)$$

Proof :

WKT $R_{xy}(\tau) = E[x(t) \cdot y(t+\tau)]$

Now, $R_{xy}(-\tau) = E[x(t) y(t-\tau)]$

Take $t-\tau = t_1 \Rightarrow t = t_1 + \tau$

$$R_{xy}(-\tau) = E[x(t_1 + \tau) y(t_1)]$$

$$= E[y(t_1) x(t_1 + \tau)]$$

$$R_{xy}(-\tau) = R_{yx}(\tau)$$

2. If $x(t)$ and $y(t)$ are WSS process, then

$$R_{xy}(\tau) \leq \sqrt{R_{xx}(0) R_{yy}(0)}$$

Proof :

By cross correlation,

$$R_{xy}(\tau) = E[x(t) y(t+\tau)]$$

By Schwartz inequality property,

$$[E(xy)]^2 \leq E(x^2) \cdot E(y^2),$$

Let $x = x(t)$, $y = y(t+\tau)$

$$E[x(t) y(t+\tau)]^2 \leq E[x^2(t)] \cdot E[y^2(t+\tau)]$$

$$[R_{xy}(\tau)]^2 \leq E[x^2(t)] E[y^2(t+\tau)]$$

$$\because E[y^2(t+\tau)] = E[y^2(t)]$$

$$[R_{xy}(\tau)]^2 \leq E[x^2(t)] \cdot E[y^2(t)]$$

By Prop. (1), $\Rightarrow E[x^2(t)] = R_{xx}(0)$

$$E[y^2(t)] = R_{yy}(0)$$

$$\therefore [R_{xy}(\tau)]^2 \leq R_{xx}(0) \cdot R_{yy}(0)$$

$$R_{xy}(\tau) \leq \sqrt{R_{xx}(0) \cdot R_{yy}(0)}$$

3. Consider two R.P. $x(t) = 3 \cos(\omega t + \theta)$ and $y(t) = 2 \cos(\omega t + \phi)$, where $\phi = \theta - \frac{\pi}{2}$ and θ is uniformly distributed random variable over $(0, 2\pi)$.

Verify that $|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0) \cdot R_{yy}(0)}$

Soln.

PDF

$$f(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \leq \theta \leq 2\pi \\ 0, & \text{otherwise} \end{cases}$$

Then $R_{xx}(t, t+\tau) = E[x(t)x(t+\tau)]$

$$= E[9 \cos(\omega t + \theta) \cos(\omega t + \omega\tau + \theta)]$$

$$= \frac{9}{2} E[2 \cos(\omega t + \omega\tau + \theta) \cdot \cos(\omega t + \theta)]$$

$$= \frac{9}{2} E[\cos \omega\tau] + \frac{9}{2} E[\cos(2\omega t + \omega\tau + 2\theta)]$$

$$= \frac{9}{2} \cos \omega\tau + \frac{9}{4\pi} \int_0^{2\pi} \cos(2\omega t + \omega\tau + 2\theta) d\theta$$

$$= \frac{9}{2} \cos \omega\tau + \frac{9}{8\pi} \left[\sin(2\omega t + \omega\tau + 2\theta) \right]_0^{2\pi}$$

$$R_{xx}(\tau) = \frac{9}{2} \cos \omega\tau$$

$$\begin{aligned}
 Y(t) &= 2 \cos(\omega t + \phi) \\
 &= 2 \cos(\omega t + \theta - \pi/2) \\
 &= 2 \cos[\pi/2 - (\omega t + \theta)]
 \end{aligned}$$

$$Y(t) = 2 \sin(\omega t + \theta)$$

$$\begin{aligned}
 \therefore R_{yy}(t, t+\tau) &= E[Y(t) Y(t+\tau)] \\
 &= 2 E[2 \sin(\omega t + \theta) \sin(\omega t + \omega\tau + \theta)] \\
 &= 2 [E[\cos \omega\tau] - E[\cos(2\omega t + \omega\tau + 2\theta)]] \\
 &= 2 \cos \omega\tau - 0 \quad \because E[\cos(2\omega t + \omega\tau + 2\theta)] = 0
 \end{aligned}$$

$$R_{yy}(\tau) = 2 \cos \omega\tau$$

$$\begin{aligned}
 R_{xy}(t, t+\tau) &= E[X(t) \cdot Y(t+\tau)] \\
 &= E[6 \cos(\omega t + \theta) \sin(\omega t + \omega\tau + \theta)] \\
 &= 3 E[2 \cos(\omega t + \theta) \sin(\omega t + \omega\tau + \theta)] \\
 &= 3 E[\sin(2\omega t + \omega\tau + 2\theta) + \sin \omega\tau] \\
 &= 3 \{ E[\sin(2\omega t + \omega\tau + 2\theta)] + E[\sin \omega\tau] \}
 \end{aligned}$$

$$R_{xy}(\tau) = 3 \sin \omega\tau$$

$$|R_{xy}(\tau)| = |3 \sin \omega\tau| \leq 3 \quad \because |\sin \omega\tau| \leq 1$$

$$\text{and } R_{xx}(0) \cdot R_{yy}(0) = \frac{9}{2} \cdot 2 = 9$$

$$\sqrt{R_{xx}(0) \cdot R_{yy}(0)} = 3$$

$$\therefore |R_{xy}(\tau)| \leq \sqrt{R_{xx}(0) \cdot R_{yy}(0)}$$

Let $x(t)$ & $y(t)$ is defined by $x(t) = A \cos \omega t + B \sin \omega t$ and $y(t) = B \cos \omega t - A \sin \omega t$ where ω is a constant and A, B are independent r.v.s. both having zero mean & variance σ^2 . Find the cross correlation of $x(t)$ & $y(t)$.
 Are $x(t)$ & $y(t)$ jointly WSS process?