Markov process

If future value depends only on the present but not the past is called Markov process.

Eg The prob of raining today depends only on the previous wether condition but does not on the past weather condition.

Markov chain

A discrete parameter Markov process is called Markov Chain.

$P(X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2} \dots x_0 = a_0) = P(X_n = a_n / X_{n-1} = a_{n-1})$

$a_1, \dots a_2$ are called states of the Markov chain

Transition probability

(i) It's satisfies the following condition

* $P_{ij} \geq 0$

* $\sum_{j = row} P_{ij} = 1$

Ex $\begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix}$

(i) $P_{ij} \geq 0$

(ii) $\frac{1}{2} + \frac{1}{2} = 1$ row

$0 + 1 = 1$

∴ The gn matrix is T.P.M

Result

① $p(x_n = a_j / x_{n-1} = a_i) = P_{ij}^{n-(n-1)} = P_{ij}^{(1)}$

② $P(x_1 = 1 / x_0 = 2) = P_{21}^{1-0} = P_{21}^{(1)}$

③ $P(x_2 = 2 / x_0 = 2) = P_{22}^{2-0} = P_{22}^{(2)}$

④ $p(x_n = a, x_{n-1} = b, x_{n-2} = c \dots x_1 = j, x_0 = k)$

$= \left(P_{ba}^{n-n+1} \; P_{cb}^{n-1-n+2} \cdots P_{kj}^{1}\right) \cdot p(x_0 = k)$

$= \left(P_{ba}^{(1)} \; P_{cb}^{(1)} \cdots P_{kj}^{(1)}\right) \cdot p(x_0 = k)$

$= (r_{ba} \quad r_{cb} \cdots r_{kj})$

(5) $p(X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2) = \left(P_{21}^{(1)} \, P_{12}^{(1)} \, P_{21}^{(1)}\right) \cdot P(X_0$

(6) $p(X_1 = 1, X_0 = 2) = \left(P_{21}^{(1)}\right) p(X_0 = 2)$

(7) $p(X_n = j) = \sum_{i=0}^{j} p(X_n = j / X_0 = i) \cdot p(X_0 = i)$

(8) $p(X_2 = 3) = \sum_{i=0}^{3} p(X_2 = 3 / X_0 = (0,1)) \cdot p(X_0 = 0,1)$

$n = 2, j = 3, i = 0,1$

$= p(X_2 = 3 / X_0 = 0) \cdot p(X_0 = 0) + p(X_2 = 3 / X_0 = 1) \cdot p(X$

$= P_{03}^{(2)} \cdot p(X_0 = 0) + P_{13}^{(2)}(X_0 = 1)$

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Transition prob of a Markov chain has three states 0, 1, 2. $P = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix}$ of a chain is

$P(X_0 = i) = \frac{1}{3}$, i varies from 0, 1, 2

(I) $P(X_3=1, X_2=2, X_1=1, X_0=2)$ (II) $P(X_2=1 / X_0=0)$

(III) $P(X_2=2)$

let $P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 3/4 & 1/4 & 0 \\ 1 & 1/4 & 1/2 & 1/4 \\ 2 & 0 & 3/4 & 1/4 \end{array}$

$P(X_0=i) = \frac{1}{3}$ $i = 0, 1, 2$

$P(X_0=0) = \frac{1}{3}$, $P(X_0=1) = \frac{1}{3}$, $P(X_0=2) = \frac{1}{3}$

(I) $P(X_3=1 / X_2=2, X_1=1, X_0=2) = \left(P_{21}^{(1)} \cdot P_{12}^{(1)} \cdot P_{21}^{(1)}\right) \cdot P(X_0=2)$

$= \left(\frac{3}{4} \cdot \frac{1}{4} \cdot \frac{3}{4}\right) \cdot \frac{1}{3} = \frac{3}{64}$

(ii) $P(X_2 = 1 / X_0 = 0) = P_{01}^{(2)} \Rightarrow$

$$P^2 = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0.62 & 0.31 & 0.06 \\ 1 & 0.31 & 0.5 & 0.18 \\ 2 & 0.18 & 0.56 & 0.25 \end{array}$$

$$P_{01}^{(2)} = 0.31$$

(iii) $P(X_2 = 2)$, $n = 2$, $j = 2$, $i = 0, 1, 2$

$$P(X_n = j) = \sum_{i=0}^{j} P(X_n = j / X_0 = i) \cdot P(X_0 = i)$$

$$= \sum_{i=0}^{2} P(X_2 = 2 / X_0 = 0, 1, 2) \cdot P(X_0 = 0, 1, 2)$$

$$= P(X_2=2/X_0=0)\cdot P(X_0=0) + P(X_2=2/X_0=1)\cdot P(X_0=1)$$

$$+ P(X_2=2/X_0=2)\cdot P(X_0=2)$$

$$= P_{02}^{(2)} P(X_0=0) + P_{12}^{(2)} P(X_0=1) + P_{22}^{(2)} P(X_0=2)$$

$$= 0.02 + 0.06 + 0.08$$

$$= 0.16$$

$$\boxed{P(X_2=2) = 0.16}$$

② The T.P.M of a Markov chain $\{X_n\}$ $n = 1, 2, 3$ having 3 states 1, 2, 3 is $P = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$ under the initial dist is $P^{(0)} = (0.7, 0.2, 0.1)$. Find (i) $P(X_2 = 3 / X_0 = 1)$ (ii) $P(X_2 = 3)$ (iii) $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$

Soln

$$P^{(0)} = (0.7, 0.2, 0.1)$$

$$(P(X_0 = 1) = 0.7,\ P(X_0 = 2) = 0.2,\ P(X_0 = 3) = 0.1\}$$

(i) $P(X_2 = 3 / X_0 = 1) = P_{13}^{(2)}$

$$P^2 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ 1 & 0.43 & 0.31 & 0.26 \\ 2 & 0.24 & 0.42 & 0.34 \\ 3 & 0.36 & 0.35 & 0.29 \end{array}$$

$$P_{13}^{(2)} = 0.26$$

$$P(X_2=3) = \sum_{i=1}^{j} P(X_2=3 / X_0=(i=1,2,3)) \cdot P(X_0= 1,2,3)$$

$n=2, j=3, i=1,2,3$

$$= P(X_2=3 / X_0=1) \cdot P(X_0=1) + P(X_2=3/X_0=2) \cdot P(X_0=2)$$

$$+ P(X_2=3/X_0=3) \cdot P(X_0=3)$$

$$= P_{13}^{(2)} \cdot P(X_0=1) + P_{23}^{(2)} \cdot P(X_0=2) + P_{33}^{(2)} P(X_0=3)$$

$$= 0.279$$

$$P(X_3=2, X_2=3, X_1=3, X_0=2) \Rightarrow$$

$$= \left(P_{32}^{(1)} \, P_{33}^{(1)} \, P_{23}^{(1)}\right) \cdot P(X_0=2)$$

$$= 0.0048$$

Irreducible chain

If $P_{ij}^{(n)} > 0$ for some n for every i and j then every state can be reached from every other state when this condition is satisfied then the Markov chain is irreducible chain.

Periodic State

$$d_i = \text{GCD} \left\{ m ; P_{ij}^{(m)} > 0 \right\}$$

if $d_i > 1$

Aperiodic state

If $d_i = 1$

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Non Null persistent

If a Markov chain is finite and irreducible

Ergodic state

Non-null persistant and A periodic

Non-Ergodic state

Non-null persistant and periodic.

(1) 3 boys A, B, C are throwing a ball to each other. A always throw a ball to B. B always throw a ball to C. But C is just as likely to throw a ball B as to A. Find the T.P.M & Classify the states.

$$P = \begin{array}{c|ccc} & A & B & C \\ \hline A & \boxed{0} & 1 & \boxed{0} \\ B & \boxed{0} & \boxed{0} & 1 \\ C & 1/2 & 1/2 & \boxed{0} \end{array}$$

$$P^2 = \begin{bmatrix} \boxed{0} & 0 & 1 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

It is finite and it is also irreducible.

∴ Non null persistant.

$$P^4 = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.25 & 0.25 & 0.5 \\ 0.25 & 0.5 & 0.25 \end{bmatrix} \qquad P^5 = \begin{bmatrix} 0.25 & 0.25 & 0.5 \\ 0.25 & 0.5 & 0.25 \\ 0.125 & 0.375 & 0.5 \end{bmatrix}$$

$$d_i = \text{Gcd}\,\{m;\ p_{ij}^{(m)} > 0\}$$

$$d_1 = \text{Gcd}\,\{3, 5, 6\} = 1$$

$$d_2 = \text{Gcd}\,\{2, 3, 4, 5, 6\} = 1$$

$$d_3 = \text{Gcd}\,\{2, 3, 4, 5, 6\} = 1$$

$$\boxed{d_i = 1}$$

$\therefore$ A periodic & non-null persistent, $\therefore$ It is Ergodic

H.W

$\begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$ Classify the states.

Soln

H.W

$$\begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$$ Classify the states.

Soln

$\therefore$ non ergodic.

If the tpm of a M.C is $\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}$ find the steady state

(i) $\pi P = \pi$

$$(\pi_1, \pi_2)\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = (\pi_1, \pi_2)$$

$$\frac{\pi_2}{2} = \pi_1 \qquad \pi_1 + \frac{\pi_2}{2} = \pi_2$$

$$\boxed{\pi_2 = 2\pi_1} \to (1) \qquad \pi_1 = \pi_2 - \frac{\pi_2}{2}$$

$$\boxed{\pi_1 = \frac{\pi_2}{2}} \to (2)$$

$$\sum_{i=1}^{n} \pi_i = 1$$

$$\pi_1 + \pi_2 = 1$$

$$\frac{\pi_2}{2} + 2\pi_1 = 1 \qquad (\because \pi_2 = 2\pi_1)$$

$$\frac{\cancel{2}\pi_1}{\cancel{2}} + 2\pi_1 = 1$$

$$3\pi_1 = 1.$$

$$\boxed{\pi_1 = 1/3}$$

$$\pi_2 = 2\pi_1$$

$$\pi_2 = 2(1/3)$$

$$\pi_2 = \frac{2}{3}$$

$\therefore$ The steady state

$$\left(\frac{1}{3} \quad \frac{2}{3}\right)$$

Steady State

If P is a T.P.M the Mar[kov] Chain $\pi = \pi_1 \ldots \pi_n$ steady State distribution

(i) $\pi P = \pi$

(ii) $\sum \pi_i = 1$