Unit - IV

Random Processes and Markov chains

* Classification
* Stationary Process
* Markov process
* Markov chains
* Transition probabilities Limiting distributions
* Birth and Death Process
* Poisson Process

Random Process:

A random process is a collection of random variables $\{X(s,t)\}$ that are functions of a real variable t where $s \in S$, S is the sample space and $t \in T$.

A Comparison between Random Variable and Random process:

Random Variable	Random Process
1. A function of the possible outcomes of an experiment. i.e., $X(s)$	A function of the possible outcomes of an experiment and also time i.e., $X(s,t)$
2. Outcome is mapped into a number 'x'	Outcomes are mapped into wave form which is a function of time 't'

Classification:

$X(t)$ \ t	Continuous	Discrete
Continuous	Continuous Random process	Continuous Random sequence
Discrete	Discrete Random process	Discrete Random sequence

Stationary Process:

A random Process is said to be stationary if its mean and variance doesn't depend on time 't'.

i.e., $E[x(t)] = \text{constant}$

& $V[x(t)] = \text{constant}$

Evolutionary Process:

A random Process that is not stationary in any sense is called as evolutionary process.

First order Stationary Process:

A random process is called 1st order stationary if its first order density function doesn't depend on time 't'.

i.e. $E[x(t)] = \text{constant}$

Wide Sense Stationary process (WSS):

A random process is said to be WSS if it satisfies

i). $E[X(t)] = \text{constant}$

ii). The Auto correlation $R_{XX}(\tau) = E[x(t)\, x(t+\tau)]$

Joint wide sense stationary Process (JWSS)

Two processes $x(t)$ & $y(t)$ are said to be JWSS if $R_{xy}(\tau) = E[x(t)\, y(t+\tau)]$

Strict sense stationary process (or) Strongly Stationary Process (SSS)

A random Process is said to be SSS, if all its statistical properties do not change with time.

Note :

Every SSS process of order 2 is a WSS process and not conversely.

---

7. The process $x(t)$ whose probability under certain conditions is given by,

$$P[x(t)=n] = \begin{cases} \dfrac{(at)^{n-1}}{(1+at)^{n-1}}, & n=1,2,\cdots \\ \dfrac{at}{1+at}, & n=0 \end{cases}$$

Show that it is not stationary [Evolutionary]

Soln. :

The probability distribution of $\{x(t)\}$ is

$x(t)=n$	0	1	2	3 ......
$P[x(t)=n]$	$\dfrac{at}{1+at}$	$\dfrac{1}{(1+at)^2}$	$\dfrac{at}{(1+at)^3}$	$\dfrac{(at)^2}{(1+at)^4}$

Mean :

$$E[x(t)=n] = \sum_{n=0}^{\infty} n\,p(n)$$

$$= 0\left(\frac{at}{1+at}\right) + 1\left(\frac{1}{(1+at)^2}\right) + 2\left(\frac{at}{(1+at)^3}\right) + \cdots\cdots$$

$$= \frac{1}{(1+at)^2}\left[1 + 2\frac{at}{1+at} + 3\left(\frac{at}{1+at}\right)^2 + \cdots\cdots\right]$$

$$= \frac{1}{(1+at)^2}\left[1 + 2\alpha + 3\alpha^2 + \cdots\cdots\right]$$

where $\alpha = \dfrac{at}{1+at}$

$$= \frac{1}{(1+at)^2}\left[1-\alpha\right]^{-2}$$

$$= \frac{1}{(1+at)^2}\left[1 - \frac{at}{1+at}\right]^{-2}$$

$$= \frac{1}{(1+at)^2}\left[\frac{1+at-at}{1+at}\right]^{-2}$$

$$= \frac{1}{(1+at)^2}(1+at)^2$$

$$E[x(t)] = 1$$

and $$E[x^2(t) = n^2] = \sum_{n=0}^{\infty} n^2 P(n)$$

$$= \sum_{n=0}^{\infty}[n(n+1)-n]\,P(n)$$

$$= \sum_{n=0}^{\infty} n(n+1)P(n) - \sum_{n=0}^{\infty} n\,P(n)$$

$$= \left[0 + 1(2)\frac{at}{(1+at)^3} + 2(3)\cancel{\frac{at}{(1+at)^3} + 3(4)}\frac{(at)^2}{(1+at)^4} + \cdots\right] - 1$$

$$= \left[0 + 1(2)\frac{1}{(1+at)^2} + 2(3)\frac{at}{(1+at)^3} + 3(4)\frac{(at)^2}{(1+at)^4} + \cdots\right] - 1$$

$$= \left[\frac{2}{(1+at)^2} + \frac{6at}{(1+at)^3} + \frac{12(at)^2}{(1+at)^4} + \cdots\right] - 1$$

$$= \frac{2}{(1+at)^2}\left[1 + \frac{3at}{1+at} + 6\frac{(at)^2}{(1+at)^2} + \cdots\right] - 1$$

$$= \frac{2}{(1+at)^2}\left[1 + 3\alpha + 6\alpha^2 + \cdots\right] - 1$$

where $\alpha = \frac{at}{1+at}$

$$= \frac{2}{(1+at)^2}[1-\alpha]^{-3} - 1$$

$$= \frac{2}{(1+at)^2}\left[1-\frac{at}{1+at}\right]^{-3} - 1$$

$$= \frac{2}{(1+at)^2}\left[\frac{1+at-at}{1+at}\right]^{-3} - 1$$

$$= \frac{2}{(1+at)^2}(1+at)^3 - 1$$

$$= 2(1+at) - 1$$

$$= 2 + 2at - 1$$

$$E[x^2(t)] = 1 + 2at$$

$$Var[x(t)] = E[x^2(t)] - \{E[x(t)]\}^2$$

$$= 1 + 2at - 1$$

$V[x(t)] = 2at$ is not a constant.

$\therefore$ The given function is not a stationary process.

Note:

$E(a) = a$; $V(a) = 0$

Formula:

1). $\cos(A+B) = \cos A \cos B - \sin A \sin B$

2). $\cos(A-B) = \cos A \cos B + \sin A \sin B$

3). $\sin(A+B) = \sin A \cos B + \cos A \sin B$

4). $\sin(A-B) = \sin A \cos B - \cos A \sin B$

5]. $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

6]. $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

7]. $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$

8]. $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

7. Show that the random process $x(t) = A\cos(\omega t + \theta)$ where A & ω are constant ; θ is uniformly distributed in $(-\pi, \pi)$ is WSS.

Soln.

Given $x(t) = A\cos(\omega t + \theta)$

To prove : WSS.

i). $E[x(t)] = $ constant

ii). $R_{XX}(\tau) = E[x(t)\, x(t+\tau)]$

Since θ is uniformly distributed in $(-\pi, \pi)$.

$$f(\theta) = \frac{1}{b-a}$$

$$= \frac{1}{\pi + \pi}$$

$$f(\theta) = \frac{1}{2\pi}$$

i). $E[x(t)] = \int x(t)\, f(\theta)\, d\theta$

$$= \int_{-\pi}^{\pi} A\cos(\omega t + \theta)\, \frac{1}{2\pi}\, d\theta$$

$$= \frac{A}{2\pi}\left[\sin(\omega t + \theta)\right]_{-\pi}^{\pi}$$

$$= \frac{A}{2\pi}\left[\sin(\pi + \omega t) - \sin(-\pi + \omega t)\right]$$

$$= \frac{A}{2\pi}\left[-\sin\omega t + \sin\omega t\right]$$

$\sin(180 + \theta) = -\sin\theta$

$E[x(t)] = 0$

$\sin(180 - \theta) = \sin\theta$

ii). $E[x(t)\,x(t+\tau)]$

$$= E[A\cos(\omega t+\theta)\; A\cos(\omega(t+\tau)+\theta)]$$

$$= A^2 E[\cos(\omega t+\theta)\cos(\omega t+\omega\tau+\theta)]$$

$$= \frac{A^2}{2} E\Big[\cos(\omega t+\theta+\omega t+\omega\tau+\theta) + \cos(\omega t+\theta-\omega t-\omega\tau-\theta)\Big]$$

$$= \frac{A^2}{2} E[\cos(2\omega t+\omega\tau+2\theta) + \cos(-\omega\tau)]$$

$$= \frac{A^2}{2}\Big\{E[\cos(2\omega t+\omega\tau+2\theta)] + E[\cos(\omega\tau)]\Big\}$$

$$= \frac{A^2}{2}\cos\omega\tau + \frac{A^2}{2} E[\cos(2\omega t+\omega\tau+2\theta)] \quad \hookrightarrow (1)$$

$E[\cos(2\omega t+\omega\tau+2\theta)]$

$$= \int_{-\pi}^{\pi} \cos(2\omega t+\omega\tau+2\theta)\frac{1}{2\pi}\, d\theta$$

$$= \frac{1}{2\pi}\left[\frac{\sin(2\omega t+\omega\tau+2\theta)}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{4\pi}\left[\sin(2\pi+2\omega t+\omega\tau) - \sin(-2\pi+2\omega t+\omega\tau)\right]$$

$$= \frac{1}{4\pi}\left[\sin(2\pi+2\omega t+\omega\tau) + \sin(2\pi-(2\omega t+\omega\tau))\right]$$

$$= \frac{1}{4\pi}\left[\sin(2\omega t+\omega\tau) - \sin(2\omega t+\omega\tau)\right]$$

$$= \frac{1}{4\pi}(0) \qquad \because \sin(360+\theta) = \sin\theta, \quad \sin(360-\theta) = -\sin\theta$$

$$= 0$$

$(1) \Rightarrow \quad E[x(t)\,x(t+\tau)] = \frac{A^2}{2}\cos\omega\tau + 0 = \frac{A^2}{2}\cos\omega\tau$

$= R_{xx}(\tau)$ which depends on τ

$\therefore$ $x(t)$ is WSS.

Q. Show that the process $x(t) = A\cos\lambda t + B\sin\lambda t$ is WSS, where A & B are random variables, if

i). $E[A] = E[B] = 0$

ii). $E[A^2] = E[B^2]$

iii). $E[AB] = 0$

soln.

Given. $x(t) = A\cos\lambda t + B\sin\lambda t$

i). $E[x(t)] = E[A\cos\lambda t + B\sin\lambda t]$

$= \cos\lambda t\, E[A] + \sin\lambda t\, E[B]$

$= \cos\lambda t\,(0) + \sin\lambda t\,(0)$ $[\because E(A) = E(B) = 0]$

$= 0$

ii). $E[x(t)\,x(t+\tau)] = E[(A\cos\lambda t + B\sin\lambda t)(A\cos\lambda(t+\tau) + B\sin\lambda(t+\tau))]$

$= E[(A\cos\lambda t + B\sin\lambda t)\ (A\cos(\lambda t+\lambda\tau) + B\sin(\lambda t+\lambda\tau))]$

$= E[A^2\cos\lambda t\cos(\lambda t+\lambda\tau) + AB\cos\lambda t\sin(\lambda t+\lambda\tau) + BA\sin\lambda t\cos(\lambda t+\lambda\tau) + B^2\sin\lambda t\sin(\lambda t+\lambda\tau)]$

$= E[A^2\cos\lambda t\cos(\lambda t+\lambda\tau)] + E[AB\cos\lambda t\sin(\lambda t+\lambda\tau)] + E[BA\sin\lambda t\cos(\lambda t+\lambda\tau)] + E[B^2\sin\lambda t\sin(\lambda t+\lambda\tau)]$

$= \cos\lambda t\cos(\lambda t+\lambda\tau)E(A^2) + \cos\lambda t\sin(\lambda t+\lambda\tau)E(AB) + \sin\lambda t\cos(\lambda t+\lambda\tau)E(BA) + \sin\lambda t\sin(\lambda t+\lambda\tau)E(B^2)$.

$= \cos\lambda t\cos(\lambda t+\lambda\tau)K + \cos\lambda t\sin(\lambda t+\lambda\tau)(0) + \sin\lambda t\cos(\lambda t+\lambda\tau)(0) + \sin\lambda t\sin(\lambda t+\lambda\tau)(K)$

$[\because E(A^2) = E(B^2) = K\ (\text{say})]$

$= K[\cos\lambda t\cos(\lambda t+\lambda\tau) + \sin\lambda t\sin(\lambda t+\lambda\tau)]$

$= K[\cos(\lambda t - (\lambda t+\lambda\tau))]$ $\quad\because \cos A\cos B + \sin A\sin B = \cos(A-B)$

$= K[\cos(-\lambda\tau)]$

$= K\cos(\lambda\tau)$ which depends on τ

$= R_{xx}(\tau)$ $\quad\therefore x(t)$ is WSS.

3]. Given a random variable y with characteristic function $\phi(\omega) = E[e^{i\omega y}]$ and a random process is defined by $x(t) = \cos(\lambda t + y)$. Show that $\{x(t)\}$ is stationary in the wide sense if $\phi(1) = \phi(2) = 0$

Soln.:

Given $\phi(\omega) = E[e^{i\omega y}]$ and $x(t) = \cos(\lambda t + y)$

Since $\phi(1) = 0$

$$\phi(1) = E[e^{iy}] = E[\cos y + i\sin y] = 0$$

$$E[\cos y] + i\,E[\sin y] = 0 + i0$$

Equating the real & imaginary parts,

$$E[\cos y] = 0 \quad \text{and} \quad E[\sin y] = 0 \rightarrow (1)$$

and $\phi(2) = 0$

$$\phi(2) = E[e^{i2y}] = E[\cos 2y + i\sin 2y] = 0$$

$$E[\cos 2y] + i\,E[\sin 2y] = 0 + i0$$

Equating the real & imaginary parts,

$$E[\cos 2y] = 0 \quad \text{and} \quad E[\sin 2y] = 0 \rightarrow (2)$$

Now, $x(t) = \cos(\lambda t + y)$

i). $E[x(t)] = E[\cos(\lambda t + y)]$

$$= E[\cos \lambda t \cos y - \sin \lambda t \sin y]$$

$$= E[\cos \lambda t \cos y] - E[\sin \lambda t \sin y]$$

$$= \cos \lambda t\, E[\cos y] - \sin \lambda t\, E[\sin y]$$

$$= \cos \lambda t\,(0) - \sin \lambda t\,(0) \quad \text{from (1)}$$

$$E[x(t)] = 0$$

ii). $E[x(t)\,x(t+\tau)] = E[\cos(\lambda t + y)\cos(\lambda(t+\tau) + y)]$

$$= E[\cos(\lambda t + y)\cos(\lambda t + \lambda\tau + y)]$$

$$= \frac{1}{2} E[\cos(\lambda t + y + \lambda t + \lambda\tau + y) + \cos(\lambda t + y - \lambda t - \lambda\tau - y)]$$

$$= \frac{1}{2} E[\cos(2\lambda t + \lambda\tau + 2y) + \cos(-\lambda\tau)]$$

$$= \frac{1}{2}\left\{ E[\cos(\underbrace{2\lambda t + \lambda\tau}_{A} + \underbrace{2y}_{B})] + E[\cos(\lambda\tau)]\right\}$$

$$= \frac{1}{2}\left\{ E[\cos(2\lambda t + \lambda\tau)\cos 2y - \sin(2\lambda t + \lambda\tau)\sin 2y] + \cos\lambda\tau\right\}$$

$$= \frac{1}{2}\left\{ E[\cos(2\lambda t + \lambda\tau)\cos 2y] - E[\sin(2\lambda t + \tau\lambda)\sin 2y] + \cos\lambda\tau\right\}$$

$$= \frac{1}{2}\left\{ \cos(2\lambda t + \lambda\tau) E[\cos 2y] - \sin(2\lambda t + \lambda\tau) E[\sin 2y] + \cos\lambda\tau\right\}$$

$$= \frac{1}{2}\left\{ 0 + 0 + \cos\lambda\tau\right\}$$

$= \dfrac{\cos\lambda\tau}{2}$ which depends on τ.

$= R_{XX}(\tau)$

$\therefore$ $X(t)$ is WSS.

Scanned with CamScanner