Problems based on auto correlation function & cross correlation function of inputs and outputs

I. A WSS process $X(t)$ with $R_{XX}(\tau) = Ae^{-a|\tau|}$ where 'A' and 'a' are real +ve constants is applied to the I/P of an LTI systems with $h(t) = e^{-bt}u(t)$ where b is a real +ve constant. Find the PSD of the o/p of the system

Soln.

Given $R_{XX}(\tau) = Ae^{-a|\tau|}$

and $h(t) = e^{-bt}u(t)$

FT of $h(t)$: $H(\omega) = F[h(t)] = \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t}\, dt$

$$= \int_{-\infty}^{\infty} e^{-bt} u(t)\, e^{-i\omega t}\, dt$$

$\because$ b is a real +ve constant in $(0, \infty)$

$$= \int_{0}^{\infty} e^{-bt} e^{-i\omega t}\, dt$$

$$= \int_0^{\infty} e^{-(b+i\omega)t} \, dt$$

$$= \left[\frac{e^{-(b+i\omega)t}}{-(b+i\omega)} \right]_0^{\infty}$$

$$= \frac{-1}{b+i\omega} (0-1)$$

$$H(\omega) = \frac{1}{b+i\omega}$$

WKT

The input power spectral density is

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) \, e^{-i\omega\tau} \, d\tau$$

$$= \int_{-\infty}^{\infty} A e^{-a|\tau|} (\cos \omega\tau - i \sin \omega\tau) \, d\tau$$

$$= A \int_{-\infty}^{\infty} e^{-a|\tau|} \cos \omega\tau \, d\tau - iA \int_{-\infty}^{\infty} e^{-a|\tau|} \sin \omega\tau \, d\tau$$

$$= 2A \int_0^{\infty} e^{-a\tau} \cos \omega\tau \, d\tau + i(0)$$

$$= 2A \frac{a}{a^2+\omega^2}$$

$$S_{XX}(\omega) = \frac{2aA}{a^2+\omega^2} \qquad \because \int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}$$

$\therefore$ The output of power spectral density is,

$$S_{yy}(\omega) = S_{xx}(\omega) * |H(\omega)|^2$$

Now

$$H(\omega) = \frac{1}{b+i\omega}$$

$$H^*(\omega) = \frac{1}{b-i\omega} \Rightarrow |H(\omega)| = \frac{1}{\sqrt{b^2+\omega^2}}$$

$$|H(\omega)|^2 = \frac{1}{b^2+\omega^2}$$

$$\therefore \quad S_{yy}(\omega) = \frac{2aA}{a^2+\omega^2} \times \frac{1}{b^2+\omega^2}$$

$$= \frac{2aA}{(a^2+\omega^2)(b^2+\omega^2)}$$

2] A system has an impulse response $h(t) = e^{-\beta t} u(t)$, find the power spectral density of the output $Y(t)$ corresponding the input $X(t)$.

Soln.

Given $h(t) = e^{-\beta t} u(t)$

Now

$$H(\omega) = F[h(t)] = \int_{-\infty}^{\infty} h(t)\, e^{-i\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-\beta t} u(t)\, e^{-i\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-\beta t}\, e^{-i\omega t}\, dt \quad \because [0,\infty] \Rightarrow u(t) = 1$$

$$= \int_0^\infty e^{-\beta t - i\omega t} \, dt$$

$$= \int_0^\infty e^{-(\beta+i\omega)t} \, dt$$

$$= \left[\frac{e^{-(\beta+i\omega)t}}{-(\beta+i\omega)} \right]_0^\infty$$

$$= \frac{-1}{\beta+i\omega} [0-1]$$

$$H(\omega) = \frac{1}{\beta+i\omega}$$

Output of PSD is,

$$S_{yy}(\omega) = S_{xx}(\omega) \, |H(\omega)|^2$$

$$= S_{xx}(\omega) \frac{1}{\beta^2+\omega^2}$$

3]. Verify the following systems are linear and also time invariant.

a). $y(t) = \alpha x(t)$

Hw b). $y(t) = t \, x(t)$

Soln.

a). Given $y(t) = \alpha x(t)$

$\Rightarrow y_1(t) = \alpha x_1(t)$ and $y_2(t) = \alpha x_2(t)$

Let $x(t) = a_1 x_1(t) + a_2 x_2(t)$

$\therefore y(x) = \alpha x(t) = \alpha [a_1 x_1(t) + a_2 x_2(t)]$

$= \alpha a_1 x_1(t) + \alpha a_2 x_2(t)$

$= a_1 [\alpha x_1(t)] + a_2 [\alpha x_2(t)]$

$\therefore\ y(x) = a_1 y_1(t) + a_2 y_2(t)$

$\Rightarrow y(t)$ is linear

Here the input $x(t)$ and the output $y(t)$ taking $x(t)$ as $x(t-t_0)$, then the output becomes,

$$y(t-t_0) = \alpha\left[x(t-t_0)\right]$$

Hence $y(t)$ is time invariant.

4]. Assume a r.p. $X(t)$ is given as input to a system with transfer function $H(\omega)=1$ for $-\omega_0 < \omega < \omega_0$. If the auto correlation function of the input process is $\frac{N_0}{2}\delta(t)$, find the auto correlation function of the output process.

Soln.

Given $\quad H(\omega) = 1, \quad -\omega_0 < \omega < \omega_0$

and $\quad R_{XX}(\tau) = \frac{N_0}{2}\delta(t)$

WKT, Input of PSD is,

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} \frac{N_0}{2}\delta(t)\, e^{-i\omega\tau}\, d\tau$$

$$= \frac{N_0}{2}\int_{-\infty}^{\infty} \delta(t)\, e^{-i\omega\tau}\, d\tau$$

$$S_{XX}(\omega) = \frac{N_0}{2}$$

Output of PSD :

$$S_{yy}(\omega) = S_{xx}(\omega) |H(\omega)|^2$$

$$S_{yy}(\omega) = \frac{N_0}{2}$$

Taking inverse FT on bothsides

$$F^{-1}[S_{yy}(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{N_0}{2} e^{i\omega\tau} d\omega$$

$$= \frac{N_0}{4\pi}\int_{-\omega_0}^{\omega_0} e^{i\omega\tau} d\omega$$

$$= \frac{N_0}{4\pi}\left[\frac{e^{i\omega\tau}}{i\tau}\right]_{-\omega_0}^{\omega_0}$$

$$= \frac{N_0}{4\pi i\tau}\left[e^{i\omega_0\tau} - e^{-i\omega_0\tau}\right]$$

$$= \frac{N_0}{4\pi i\tau} 2i \sin\omega_0\tau$$

$$R_{yy}(\tau) = \frac{N_0}{2\pi\tau}\sin\omega_0\tau$$

5]. $X(t)$ is the input voltage to a circuit (system) and $y(t)$ is the output voltage. $\{x(t)\}$ is a stationary random process with $\mu_x = 0$ and $R_{xx}(\tau) = e^{-\alpha|\tau|}$, find μ_y, $S_{yy}(\omega)$, if the power transfer function is $H(\omega) = \frac{R}{R+iL\omega}$

Soln.

WKT $\quad Y(t) = \int_{-\infty}^{\infty} h(u)\, X(t-u)\, du$

$$E[Y(t)] = \int_{-\infty}^{\infty} h(u)\, E[X(t-u)]\, du$$

$$\because E[X(t-u)] = 0$$

$$= 0$$

$\Rightarrow \quad \mu_Y = 0$

Input of PSD:

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\alpha|\tau|} (\cos \omega\tau - i \sin \omega\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\alpha|\tau|} \cos \omega\tau\, d\tau - i \int_{-\infty}^{\infty} e^{-\alpha|\tau|} \sin \omega\tau\, d\tau$$

$$= 2\int_{0}^{\infty} e^{-\alpha\tau} \cos \omega\tau\, d\tau - i(0)$$

$$S_{XX}(\omega) = 2 \frac{\alpha}{\alpha^2+\omega^2}$$

$$S_{XX}(\omega) = \frac{2\alpha}{\alpha^2+\omega^2}$$

Output of PSD:

$$S_{YY}(\omega) = S_{XX}(\omega)\, |H(\omega)|^2$$

$$= \frac{2\alpha}{\alpha^2+\omega^2} \frac{R^2}{R^2+L^2\omega^2}$$

$$= \frac{2\alpha}{\alpha^2+\omega^2}\left[\frac{R^2/L^2}{\frac{R^2}{L^2}+\omega^2}\right]$$

$$= \frac{2\alpha}{\alpha^2+\omega^2}\left[\frac{\beta^2}{\beta^2+\omega^2}\right]$$

$$S_{yy}(\omega) = \frac{2\alpha\beta^2}{(\alpha^2+\omega^2)(\beta^2+\omega^2)} \rightarrow (1)$$

Apply partial fraction method,

$$\frac{1}{(\alpha^2+\omega^2)(\beta^2+\omega^2)} = \frac{A}{\alpha^2+\omega^2}+\frac{B}{\beta^2+\omega^2} \rightarrow (2)$$

$$1 = A(\beta^2+\omega^2)+B(\alpha^2+\omega^2)$$

Put $\omega^2 = -\beta^2 \Rightarrow 1 = B(\alpha^2-\beta^2)$

$$B = \frac{-1}{\beta^2-\alpha^2}$$

Put $\omega^2 = -\alpha^2 \Rightarrow 1 = A(\beta^2-\alpha^2)$

$$A = \frac{1}{\beta^2-\alpha^2}$$

$$(2) \Rightarrow \frac{1}{(\alpha^2+\omega^2)(\beta^2+\omega^2)} = \frac{\frac{1}{\beta^2-\alpha^2}}{\alpha^2+\omega^2} - \frac{\frac{1}{\beta^2-\alpha^2}}{\beta^2+\omega^2}$$

$$= \frac{1}{\beta^2-\alpha^2}\left[\frac{1}{\alpha^2+\omega^2}-\frac{1}{\beta^2+\omega^2}\right]$$

$$(1) \Rightarrow S_{yy}(\omega) = 2\alpha\beta^2\frac{1}{\beta^2-\alpha^2}\left[\frac{1}{\alpha^2+\omega^2}-\frac{1}{\beta^2+\omega^2}\right]$$

$$= \frac{2\alpha\beta^2}{(\beta^2-\alpha^2)(\alpha^2+\omega^2)} - \frac{2\alpha\beta^2}{(\beta^2-\alpha^2)(\beta^2+\omega^2)}$$

$$= \frac{\beta^2}{\beta^2-\alpha^2}\left[\frac{2\alpha}{\alpha^2+\omega^2}\right] - \frac{\alpha\beta}{\beta^2-\alpha^2}\left[\frac{2\beta}{\beta^2+\omega^2}\right]$$

Taking Inverse FT on bothsides,

$$F^{-1}[S_{yy}(\omega)] = \frac{\beta^2}{\beta^2-\alpha^2} F^{-1}\left[\frac{2\alpha}{\alpha^2+\omega^2}\right] - \frac{\alpha\beta}{\beta^2-\alpha^2} F^{-1}\left[\frac{2\beta}{\beta^2+\omega^2}\right]$$

$$R_{yy}(\tau) = \frac{\beta^2}{\beta^2-\alpha^2} e^{-\alpha|\tau|} - \frac{\alpha\beta}{\beta^2-\alpha^2} e^{-\beta|\tau|}$$

$$= \frac{R^2/L^2}{\frac{R^2}{L^2}-\alpha^2} e^{-\alpha|\tau|} - \frac{\alpha\frac{R}{L}}{\frac{R^2}{L^2}-\alpha^2} e^{-\beta|\tau|}$$

$$= \lambda e^{-\alpha|\tau|} - u e^{-\beta|\tau|}$$

where $\lambda = \dfrac{R^2/L^2}{\frac{R^2}{L^2}-\alpha^2}$

$$u = \frac{\alpha R/L}{R^2/L^2-\alpha^2}$$