$$P(X<Y) = \frac{1}{8}\int_{-x}^{x}\left[\frac{y^3}{3}-\frac{y^3}{2}\right]dy$$

$$= \frac{1}{8}\int_{-x}^{x}\left[\frac{2y^3-3y^3}{6}\right]dy$$

$$= \frac{-1}{48}\int_{-x}^{x} y^3\,dy$$

$$= \frac{-1}{48}\left[\frac{y^4}{4}\right]_{-x}^{x} = \frac{-1}{192}\left[x^4 - x^4\right] = 0$$

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7) If the joint p.d.f X and Y is given by,

$$f(x,y) = \begin{cases} K(6-x-y) & , 0<x<2, 2<y<4 \\ 0 & , \text{otherwise.} \end{cases}$$

(i) Find K (ii) $P(X<1 \cap Y<3)$ (iii) $P(X<1 / Y<3)$

(iv) $P(X+Y<3)$.

Solution:

(i) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\int_{0}^{2}\int_{2}^{4} K(6-x-y)\,dx\,dy = 1$$

$$K\int_{0}^{2}\left[6x-\frac{x^2}{2}-xy\right]_2^4 dy = 1$$

$$K\int_{0}^{2}\left[24-12-\frac{16}{2}+\frac{4}{2}-4y+2y\right]dy = 1$$

$$K\int_{0}^{2}(6-2y)\,dy = 1$$

$$c\int_0^2 \left[x^3 - \cancel{\frac{x^3}{2}} + x^3 + \cancel{\frac{x^3}{2}}\right] dx = 1$$

$$2c\int_0^2 x^3\, dx = 1$$

$$2c\left[\frac{x^4}{4}\right]_0^2 = 1$$

$$\frac{2c}{4}\cdot 2^4 = 1 \Rightarrow 8c = 1 \Rightarrow \boxed{c = \frac{1}{8}}$$

(iii) $f(x,y) = \begin{cases} \frac{1}{8}x(x-y), & 0<x<2,\ -x<y<x \\ 0, & \text{otherwise.} \end{cases}$

(ii) Marginal pdf of 'x' is,

$$f_X(x) = f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{-x}^{x} \frac{1}{8}x(x-y)\,dy$$

$$= \frac{1}{8}\int_{-x}^{x}(x^2 - xy)\,dy$$

$$= \frac{1}{8}\left[x^2y - \frac{xy^2}{2}\right]_{-x}^{x}$$

$$= \frac{1}{8}\left[x^3 - \cancel{\frac{x^3}{2}} + x^3 + \cancel{\frac{x^3}{2}}\right]$$

$$= \frac{1}{8}\times 2x^3$$

$$\boxed{f(x) = \frac{x^3}{4}} \text{ where } 0<x<2$$

(iv) Marginal pdf of 'y' is,

$$f_y(y) = f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^2 \frac{1}{8} x(x-y)\,dx$$

$$= \frac{1}{8}\left[\frac{x^3}{3} - y\frac{x^2}{2}\right]_0^2$$

$$= \frac{1}{8}\left[\frac{2^3}{3} - y\cdot\frac{2^2}{2}\right] = \frac{1}{8}\left[\frac{8}{3} - 2y\right] = \frac{1}{3} - \frac{y}{4}$$

$$\boxed{f(y) = \frac{1}{3} - \frac{y}{4}}$$ where $-x < y < x$

(iii) $$f(y|x) = \frac{f(x,y)}{f(x)}$$

$$= \frac{\frac{1}{8}x(x-y)}{\frac{x^3}{4}} = \frac{1}{2x^2}(x-y)$$

$$\boxed{f(y/x) = \frac{x-y}{2x^2}}$$

(v) $$P(X<Y) = \int_{-x}^{x}\int_0^{y} f(x,y)\,dx\,dy$$

$$= \int_{-x}^{x}\int_0^{y} \frac{1}{8} x(x-y)\,dx\,dy$$

$$= \frac{1}{8}\int_{-x}^{x}\left[\frac{x^3}{3} - y\frac{x^2}{2}\right]_0^{y} dy$$

(17)

(iii)
$$= \frac{1}{8}\left[\frac{y^3}{6} - \frac{6y^2}{2} + \frac{27}{2}y\right]_2^3$$

$$= \frac{1}{8}\left[\frac{27}{6} - \frac{8}{6} - 3(9-4) + \frac{27}{2}(3-2)\right]$$

$$= \frac{1}{8}\left[\frac{27}{6} - \frac{8}{6} - 15 + \frac{27}{2}\right] = \frac{1}{8}\left[\frac{19}{6} - \frac{3}{2}\right]$$

$$= \frac{1}{8}\left[\frac{19-9}{6}\right] = \frac{1}{8} \times \frac{10}{6} = \frac{5}{24}$$

$$\boxed{P(X+Y<3) = \frac{5}{24}}$$

(8) If the joint distribution function of X and Y is given by.

$$F(x,y) = (1-e^{-x})(1-e^{-y}) \text{ for } x>0, y>0$$
$$= 0 \quad \text{, otherwise}$$

(i) Find the marginal densities of X and Y.

(ii) Are X and Y independent.

(iii) $P(1<X<3, 1<Y<2)$

Solution:

Given: $F(x,y) = (1-e^{-x})(1-e^{-y})$
$$= 1 - e^{-x} - e^{-y} + e^{-(x+y)}$$

The joint pdf is given by,

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x\, \partial y}$$

$$= \frac{\partial^2}{\partial x\, \partial y}\left[1 - e^{-x} - e^{-y} + e^{-(x+y)}\right]$$

$$= \frac{\partial}{\partial x}\left[0 - 0 + e^{-y} - e^{-(x+y)}\right]$$