PROBLEMS :

1) The joint pdf of random variable X and Y is given by $f(x,y) = Kxy\,e^{-(x^2+y^2)}$, $x>0$, $y>0$. Find the value of k and prove also that X and Y are independent.

Solution :

Given : $f(x,y)$ is a p.d.f

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$$

$$\therefore \int_{0}^{\infty}\int_{0}^{\infty} Kxy\,e^{-(x^2+y^2)}\,dy\,dx = 1 \qquad [\because x>0, y>0]$$

$$K\int_{0}^{\infty} y e^{-y^2}\,dy \cdot \int_{0}^{\infty} x e^{-x^2}\,dx = 1 \longrightarrow (1)$$

Consider $\displaystyle\int_{0}^{\infty} x e^{-x^2}\,dx = \int_{0}^{\infty} e^{-t}\,\frac{dt}{2}$

put $x^2 = t$

$2x\,dx = dt$

$x = 0,\ t = 0$

$x = \infty,\ t = \infty$

$$= \frac{1}{2}\left[\frac{e^{-t}}{-1}\right]_0^{\infty}$$

$$= \frac{1}{2}$$

$\therefore$ (1) becomes,

$$K\cdot\frac{1}{2}\cdot\frac{1}{2} = 1 \Rightarrow \frac{k}{4} = 1$$

$$\boxed{K = 4}$$

$$f(x,y) = 4xy\,e^{-(x^2+y^2)}$$

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To prove X and Y are independent :

$$f(x)\cdot f(y) = f(x,y)$$

Now, $f(x) = f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$$= \int_{0}^{\infty} 4xye^{-(x^2+y^2)}\,dy$$

$$= 4xe^{-x^2}\int_{0}^{\infty} ye^{-y^2}\,dy$$

$$= 4xe^{-x^2}\cdot\frac{1}{2}$$

$$f(x) = 2xe^{-x^2}$$

III^{ly} $f(y) = 2ye^{-y^2}$

$$\therefore f(x)\cdot f(y) = (2xe^{-x^2})(2ye^{-y^2})$$

$$= 4xye^{-(x^2+y^2)}$$

$$\boxed{f(x)\,f(y) = f(x,y)}$$

$\therefore$ X & Y are independent

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2) If the joint pdf of a random variable (X,Y) is given by,

$$f(x,y) = \begin{cases} x^2 + \dfrac{xy}{3} & ; 0 \le x \le 1,\ 0 \le y \le 2 \\ 0 & ; \text{otherwise} \end{cases}$$

Find (i) $P(X > 1/2)$ (ii) $P(Y < X)$ (iii) $P(Y < 1/2 \,/\, X < \frac{1}{2})$

(iv) Check whether the conditional densities of X on Y and Y on X are valid.

Solution :

Given : $f(x,y) = x^2 + \frac{xy}{3}$, $0 \le x \le 1$, $0 \le y \le 2$

(i) $$P(X > 1/2) = \int_0^2 \int_{1/2}^1 \left(x^2 + \frac{xy}{3}\right) dx\, dy$$

$$= \int_0^2 \left[\frac{x^3}{3} + \frac{y}{3}\frac{x^2}{2}\right]_{1/2}^1 dy$$

$$= \int_0^2 \left(\frac{1}{3} + \frac{y}{6} - \frac{1}{24} - \frac{y}{24}\right) dy$$

$$= \left[\frac{y}{3} + \frac{y^2}{12} - \frac{y}{24} - \frac{y^2}{24 \times 2}\right]_0^2$$

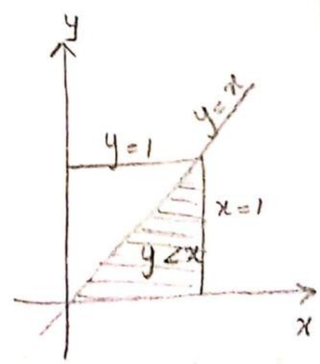
$$= \frac{2}{3} + \frac{4}{12} - \frac{2}{24} - \frac{4}{48}$$

$$= \frac{2}{3} + \frac{1}{3} - \frac{1}{12} - \frac{1}{12}$$

$$= \frac{8+4-1-1}{12} = \frac{10}{12} = \frac{5}{6}$$

$$\boxed{P\left(X > \frac{1}{2}\right) = \frac{5}{6}}$$

(ii) $$P(Y < X) = \int_0^1 \int_0^x \left(x^2 + \frac{xy}{3}\right) dx\, dy$$



$$= \int_0^1 \left[\frac{x^3}{3} + \frac{y}{3}\frac{x^2}{2}\right]_0^x dy$$

$$= \int_0^1 \left(\frac{x^3}{3} + \frac{y}{6}x^2\right) dy$$

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$$\cancel{P(Y<X) = \left[\frac{x^3 y}{3} + \frac{x^2}{6}\cdot\frac{y^2}{2}\right]_0^1}$$

$$\cancel{= \frac{x^3}{3} + \frac{x^2}{12}}$$

$$P(Y<X) = \int_0^1 \int_0^x \left(x^2 + \frac{xy}{3}\right) dy\, dx$$

$$= \int_0^1 \left[x^2 y + \frac{x}{3}\cdot\frac{y^2}{2}\right]_0^x dx$$

$$= \int_0^1 \left[x^3 + \frac{x^3}{6}\right] dx$$

$$= \left[\frac{x^4}{4} + \frac{x^4}{24}\right]_0^1 = \frac{1}{4} + \frac{1}{24} = \frac{6+1}{24}$$

$$\boxed{P(Y<X) = \frac{7}{24}}$$

(iii) $$P\left(Y<\tfrac{1}{2} / X<\tfrac{1}{2}\right) = \frac{P\left(X<\frac{1}{2}, Y<\frac{1}{2}\right)}{P\left(X<\frac{1}{2}\right)}$$

$$P\left(X<\tfrac{1}{2}, Y<\tfrac{1}{2}\right) = \int_0^{1/2}\int_0^{1/2} f(x,y)\, dx\, dy$$

$$= \int_0^{1/2}\int_0^{1/2} \left(x^2 + \frac{xy}{3}\right) dx\, dy$$

$$= \int_0^{1/2} \left[x^2 y + \frac{x}{3}\,\frac{y^2}{2}\right]_0^{1/2} dx$$

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