Continuous Two Dimensional Random Variables

1. Given $F(x,y) = \begin{cases} cx(x-y), & 0<x<2,\ -x<y<x \\ 0, & \text{otherwise} \end{cases}$

Find i). $c$ ii). $F(x)$ iii). $F(y/x)$

Soln.

i). $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,y)\,dy\,dx = 1$

$$\int_0^2\int_{-x}^{x} cx(x-y)\,dy\,dx = 1$$

$$c\int_0^2\int_{-x}^{x} (x^2-xy)\,dy\,dx = 1$$

$$c\int_0^2 \left[x^2y - x\frac{y^2}{2}\right]_{y=-x}^{x} dx = 1$$

$$c\int_0^2 \left[\left(x^3-\frac{x^3}{2}\right)-\left(-x^3-\frac{x^3}{2}\right)\right] dx = 1$$

$$c\int_0^2 \left[x^3-\frac{x^3}{2}+x^3+\frac{x^3}{2}\right] dx = 1$$

$$c\int_0^2 2x^3\,dx = 1$$

$$2c\left(\frac{x^4}{4}\right)_0^2 = 1$$

$$\frac{c}{2}(2^4-0) = 1$$

$$\frac{16c}{2} = 1$$

$$8c = 1 \Rightarrow c = 1/8$$

$$\therefore f(x,y) = \begin{cases} \frac{1}{8}x(x-y), & 0<x<2,\ -x<y<x \\ 0, & \text{otherwise.} \end{cases}$$

ii). Marginal density function of X (MDF of X)

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{-x}^{x} \frac{1}{8}x(x-y)\,dy$$

$$= \frac{1}{8}\int_{-x}^{x} (x^2 - xy)\,dy$$

$$= \frac{1}{8}\left[x^2 y - x\frac{y^2}{2}\right]_{y=-x}^{x}$$

$$= \frac{1}{8}\left[\left(x^3 - \frac{x^3}{2}\right) - \left(-x^3 - \frac{x^3}{2}\right)\right]$$

$$= \frac{1}{8}\left[x^3 - \frac{x^3}{2} + x^3 + \frac{x^3}{2}\right]$$

$$= \frac{2x^3}{8}$$

$$f(x) = \frac{x^3}{4}, \quad 0<x<2$$

iii). $f(Y/X)$

WKT $$f(Y/X) = \frac{f(X,Y)}{f(X)}$$

$$= \frac{\frac{1}{8}x(x-y)}{x^3/4}$$

$$= \frac{4/8\,(x^2 - xy)}{x^3}$$

$$= \frac{1}{2}\,\frac{x(x-y)}{x^3}$$

$$F(y/x) = \frac{x-y}{2x^2}$$

2]. The joint probability density function

$$F(x,y) = \begin{cases} xy^2 + \frac{x^2}{8}, & 0 \le x \le 2 \ \& \ 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

i). $P(X>1/Y<1/2)$ ii). $P(Y<1/2 \,/\, X>1)$

iii). $P(X<Y)$ iv). $P(X+Y\le 1)$

Soln.

i). $P(X>1/Y<\frac{1}{2})$

$$= \frac{P(X>1,\ Y<\frac{1}{2})}{P(Y<\frac{1}{2})} \rightarrow (1)$$

Now, $P(X>1,\ Y<\frac{1}{2}) = \int_1^2\int_0^{1/2} (xy^2 + \frac{x^2}{8})\,dy\,dx$

$$= \int_1^2 \left[ x\frac{y^3}{3} + \frac{x^2}{8}y \right]_{y=0}^{1/2} dx$$

$$= \int_1^2 \left\{ \frac{x}{3}\left(\frac{1}{8}\right) + \frac{x^2}{8}\left(\frac{1}{2}\right) - 0 \right\} dx$$

$$= \int_1^2 \left( \frac{x}{24} + \frac{x^2}{16} \right) dx$$

$$= \left( \frac{1}{24}\,\frac{x^2}{2} + \frac{1}{16}\,\frac{x^3}{3} \right)_1^2$$

$$= \left( \frac{4}{48} + \frac{8}{48} \right) - \left( \frac{1}{48} + \frac{1}{48} \right)$$

$$= \frac{12}{48} - \frac{2}{48}$$

$$= \frac{10}{48}$$

$$= \frac{5}{24} \rightarrow (2)$$

ii). $P\left(Y < \frac{1}{2} \,/\, X > 1\right)$

$$= \frac{P(X > 1,\ Y < 1/2)}{P(X > 1)}$$

$$P\left(Y < \frac{1}{2}\right) = \int_0^2 \int_0^{1/2} F(x, y)\, dy\, dx$$

$$= \int_0^2 \int_0^{1/2} \left( xy^2 + \frac{x^2}{8} \right) dy\, dx$$

$$= \int_0^2 \left[ \frac{xy^3}{3} + \frac{x^2}{8}\, y \right]_{y=0}^{1/2} dx$$

$$= \int_0^2 \left[ \frac{x}{3}\left(\frac{1}{8}\right) + \frac{x^2}{8}\left(\frac{1}{2}\right) \right] dx$$

$$= \int_0^2 \left[ \frac{x}{24} + \frac{x^2}{16} \right] dx$$

$$= \left[ \frac{x^2}{48} + \frac{x^3}{48} \right]_0^2$$

$$= \frac{4}{48} + \frac{8}{48} = \frac{12}{48}$$

$$(1) \Rightarrow P(X>1 / Y<\tfrac{1}{2}) = \frac{5}{24} \cdot \frac{4}{1}$$

$$= \frac{5}{6}$$

ii). $P(Y<\frac{1}{2} / X>1)$

$$= \frac{P(X>1, Y<\frac{1}{2})}{P(X>1)} \rightarrow (3)$$

Now

$$P(X>1) = \int_1^2 \int_0^1 f(x,y)\, dy\, dx$$

$$= \int_1^2 \int_0^1 \left(xy^2 + \frac{x^2 y}{8}\right) dy\, dx$$

$$= \int_1^2 \left[\frac{xy^3}{3} + \frac{x^2}{8}\frac{y^2}{2}\right]_{y=0}^{1} dx$$

$$= \int_1^2 \left[\frac{x}{3} + \frac{x^2}{8}\right] dx$$

$$= \left(\frac{x^2}{6} + \frac{x^3}{24}\right)_1^2$$

$$= \left(\frac{4}{6} + \frac{8}{24}\right) - \left(\frac{1}{6} + \frac{1}{24}\right)$$

$$= \frac{16+8-4-1}{24}$$

$$= \frac{19}{24}$$

$$(3) \Rightarrow P(Y<\tfrac{1}{2} / X>1) = \frac{5}{24} \times \frac{24}{19} = \frac{5}{19}$$

ii i). $P(X < Y)$

$$= \int_0^1 \int_0^y f(x, y)\, dx\, dy$$

$$= \int_0^1 \int_0^y \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$= \int_0^1 \left[ \frac{x^2}{2} y^2 + \frac{x^3}{24} \right]_0^y dy$$

$$= \int_0^1 \left[ \frac{y^4}{2} + \frac{y^3}{24} \right] dy$$

$$= \left[ \frac{y^5}{10} + \frac{y^4}{96} \right]_0^1$$

$$\frac{24 \times 4}{96}$$

$$= \frac{1}{10} + \frac{1}{96}$$

$$= \frac{96+10}{960} = \frac{106}{960}$$

iii). $P(X + Y \leq 1)$

$$= \int_0^1 \int_0^{1-y} f(x, y)\, dx\, dy$$

$$= \int_0^1 \int_0^{1-y} \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$= \int_0^1 \left[ \frac{x^2}{2} y^2 + \frac{x^3}{24} \right]_{x=0}^{1-y} dy$$

$$= \int_0^1 \left[ \frac{(1-y)^2 y^2}{2} + \frac{(1-y)^3}{24} \right] dy$$

$$= \int_0^1 \left[ \frac{(1+y^2-2y)\,y^2}{2} + \frac{(1-y)\,(1-y)^2}{24} \right] dy$$

$$= \int_0^1 \left[ \frac{y^2+y^4-2y^3}{2} + \frac{(1-y)\,(1+y^2-2y)}{24} \right] dy$$

$$= \int_0^1 \left[ \frac{y^2+y^4-2y^3}{2} + \frac{1+y^2-2y-y-y^3+2y^2}{24} \right] dy$$

$$= \frac{1}{24} \int_0^1 \left[ 12(y^2+y^4-2y^3) + (1+3y^2-3y-y^3) \right] dy$$

$$= \frac{1}{24} \int_0^1 \left[ 12y^2+12y^4-24y^3+1+3y^2-3y-y^3 \right] dy$$

$$= \frac{1}{24} \int_0^1 \left[ 12y^4-25y^3+15y^2-3y+1 \right] dy$$

$$= \frac{1}{24} \left[ \frac{12y^5}{5} - 25\frac{y^4}{4} + 15\frac{y^3}{3} - 3\frac{y^2}{2} + y \right]_{y=0}^{1}$$

$$= \frac{1}{24} \left[ \left( \frac{12}{5} - \frac{25}{4} + \frac{15}{3} - \frac{3}{2} + 1 \right) - 0 \right]$$

$$2 \,\underline{|\, 5,\ 4,\ 3,\ 2}$$
$$\quad 5,\ 2,\ 3,\ 1$$

$$= \frac{1}{24} \left[ \frac{144-375+300-90+60}{60} \right]$$

$$\frac{20\times 3}{60}$$

$$= \frac{39}{1440}$$

$$= 0.027$$

3] The joint PDF of the RV is given by,

$f(x, y) = kxy\, e^{-(x^2+y^2)}, \quad x>0, y>0.$

find i). k ii). check x & y are independent.

Soln.

i). $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dy\, dx = 1$$

$$\int_{0}^{\infty}\int_{0}^{\infty} kxy\, e^{-(x^2+y^2)}\, dy\, dx = 1$$

$$k\int_{0}^{\infty}\int_{0}^{\infty} xy\, e^{-x^2}\, e^{-y^2}\, dy\, dx = 1$$

Take $x^2 = s$ | $y^2 = t$

$ds = 2x\,dx$ | $2y\,dy = dt$

$\frac{ds}{2} = x\,dx$ | $y\,dy = \frac{dt}{2}$

Now,

$$k\int_{0}^{\infty}\int_{0}^{\infty} e^{-s}\, e^{-t}\, \frac{dt}{2}\, \frac{ds}{2} = 1$$

$$\frac{k}{4}\int_{0}^{\infty}\int_{0}^{\infty} e^{-s}\, e^{-t}\, dt\, ds = 1$$

$$\frac{k}{4}\int_{0}^{\infty} e^{-s}\left[\frac{e^{-t}}{-1}\right]_{0}^{\infty} ds = 1$$

$$-\frac{k}{4}\int_{0}^{\infty} e^{-s}\,[0-1]\, ds = 1$$

$$\frac{k}{4}\left(\frac{e^{-s}}{-1}\right)_{0}^{\infty} = 1$$

$$\frac{-K}{4}(0-1) = 1$$

$$\frac{K}{4} = 1$$

$$\boxed{K = 4}$$

ii). $X$ & $Y$ are independent.

To prove $F(x, y) = F(x) \cdot F(y)$

Now,

$$F(x) = \int_0^\infty F(x, y)\, dy$$

$$= \int_0^\infty 4xy\, e^{-(x^2+y^2)}\, dy$$

$$= 4x e^{-x^2} \int_0^\infty y\, e^{-y^2}\, dy$$

Take $y^2 = t$

$2y\,dy = dt$

$y\,dy = \frac{dt}{2}$

$$= 4x e^{-x^2} \int_0^\infty e^{-t}\, \frac{dt}{2}$$

$$= 2x e^{-x^2} \left[\frac{e^{-t}}{-1}\right]_0^\infty$$

$$= -2x e^{-x^2}(0-1)$$

$$F(x) = 2x e^{-x^2}, \quad x > 0$$

$$F(y) = \int_0^\infty F(x, y)\, dx$$

$$= \int_0^\infty 4xy\, e^{-(x^2+y^2)}\, dx$$

$$= 4y e^{-y^2} \int_0^\infty x e^{-x^2}\, dx$$

Take $x^2 = s$

$2x\, dx = ds$

$x\, dx = \frac{ds}{2}$

$$= 4y e^{-y^2} \int_0^\infty e^{-s} \frac{ds}{2}$$

$$= 2y e^{-y^2} \left[ \frac{e^{-s}}{-1} \right]_0^\infty$$

$$= -2y e^{-y^2} (0-1)$$

$f(y) = 2y e^{-y^2}, \quad y > 0$

$\therefore f(x) \cdot f(y) = 2x e^{-x^2} \cdot 2y e^{-y^2}$

$= 4xy\, e^{-(x^2+y^2)}$

$= f(x, y)$

$\therefore$ $x$ & $y$ are independent.

4]. If the joint density function of $x$ & $y$ is given by,

$$F(x, y) = \begin{cases} (1-e^{-x})(1-e^{-y}), & x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

To prove $x$ & $y$ are independent.

Soln.

WKT $\quad f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$

$$= \frac{\partial^2}{\partial x \partial y}\left[1 - e^{-x} - e^{-y} + e^{-(x+y)}\right]$$

$$= \frac{\partial}{\partial x}\left[-e^{-y}(-1) + e^{-x} e^{-y}(-1)\right]$$

$$= \frac{\partial}{\partial x}\left[e^{-y} - e^{-x} e^{-y}\right]$$

$$= 0 - e^{-y} e^{-x}(-1)$$

$$= e^{-x} e^{-y}$$

$$f(x,y) = e^{-(x+y)}$$

To prove:

$$f(x,y) = f(x) \cdot f(y)$$

Now,

$$f(x) = \int_0^\infty e^{-(x+y)}\, dy$$

$$= e^{-x} \int_0^\infty e^{-y}\, dy$$

$$= e^{-x}\left(\frac{e^{-y}}{-1}\right)_0^\infty$$

$$= -e^{-x}[0-1]$$

$$f(x) = e^{-x}$$

$$f(y) = \int_0^\infty e^{-(x+y)}\, dx$$

$$= \int_0^\infty e^{-x} e^{-y}\, dx$$

$$= e^{-y}\left(\frac{e^{-x}}{-1}\right)_0^\infty$$

$$= -e^{-y}(0-1)$$

$$F(y) = e^{-y}$$

$$F(x) \cdot F(y) = e^{-x} \cdot e^{-y}$$

$$= e^{-(x+y)}$$

$$= F(x, y)$$

$\therefore$ $x$ and $y$ are independent.