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GAMMA DISTRIBUTION (or Erlang Distribution)

One parameter form:

A continuous random variable X is said to follow Gamma distribution with parameter, if its p.d.f is

$$f(x) = \begin{cases} \dfrac{e^{-x} x^{\lambda-1}}{\Gamma_\lambda}, & \lambda > 0,\ 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

Moment generating function:

$$M_X(t) = E(e^{tx})$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_0^{\infty} e^{tx} \frac{e^{-x} x^{\lambda-1}}{\Gamma_\lambda}\,dx$$

$$\Gamma_n = \int_0^{\infty} e^{-x} x^{n-1}\,dx$$

$$= \frac{1}{\Gamma_\lambda} \cdot \int_0^{\infty} e^{(t-1)x} x^{\lambda-1}\,dx$$

$$= \frac{1}{\Gamma_\lambda} \int_0^{\infty} e^{-(1-t)x} x^{\lambda-1}\,dx$$

Put $(1-t)x = y$

$(1-t)dx = dy$

If $x = 0,\ y = 0$

$x = \infty,\ y = \infty$

$$= \frac{1}{\Gamma_\lambda} \int_0^{\infty} e^{-y} \left(\frac{y}{1-t}\right)^{\lambda-1} \frac{dy}{(1-t)}$$

$$= \frac{1}{\Gamma_\lambda (1-t)^\lambda} \int_0^{\infty} e^{-y} y^{\lambda-1}\,dy$$

$$= \frac{1}{\Gamma_\lambda} \cdot \frac{\Gamma_\lambda}{(1-t)^\lambda}$$

$$= (1-t)^{-\lambda}$$

$$\therefore \; M_X(t) = (1-t)^{-\lambda} \quad \text{where } |t| < 1$$

$$\boxed{M_X(t) = 1 + \lambda t + \frac{\lambda(\lambda+1)}{2!} t^2 + \cdots}$$

Mean and Variance :

$$\text{Mean} = \mu_1' = \text{Coefficient of } \frac{t}{1!}$$
$$= \lambda$$

$$\therefore \boxed{\text{Mean} = \lambda}$$

$$\mu_2' = \text{Coefficient of } \frac{t^2}{2!}$$
$$= \lambda(\lambda+1)$$

$$\text{Variance} = \mu_2' - (\mu_1')^2$$
$$= \lambda(\lambda+1) - \lambda^2$$
$$= \lambda^2 + \lambda - \lambda^2$$

$$\boxed{\text{Variance} = \lambda}$$

$$M_X(t) = (1-t)^{-\lambda}$$
$$M_X'(t) = (-\lambda)(1-t)^{-\lambda-1}(-1)$$
$$= \lambda(1-t)^{-\lambda-1}$$
$$M_X'(0) = \lambda$$
$$\boxed{\text{Mean} = \lambda}$$
$$M_X''(t) = (+\lambda)(-\lambda-1)(1-t)^{-\lambda-2}(-1)$$
$$= +\lambda(\lambda+1)(1-t)^{-\lambda-2}$$
$$M_X''(0) = +\lambda(\lambda+1)$$
$$\text{Variance} = E(X^2) - [E(X)]^2$$
$$= +\lambda(\lambda+1) - \lambda^2$$
$$= +\lambda^2 + \lambda - \lambda^2$$

$$\therefore \boxed{\text{Mean} = \text{Variance} = \lambda}$$

$$\boxed{\text{Variance} = \lambda}$$

Characteristic function :

$$\varphi_X(t) = E(e^{itx})$$
$$= \int_{-\infty}^{\infty} e^{itx} f(x)\, dx$$
$$= \int_0^{\infty} e^{itx} \frac{e^{-x} x^{\lambda-1}}{\Gamma_\lambda}\, dx$$

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$$\phi_X(t) = \frac{1}{\Gamma_\lambda} \int_0^\infty e^{-(1-it)x} x^{\lambda-1}\, dx$$

$$= \frac{1}{\Gamma_\lambda} \frac{\Gamma_\lambda}{(1-it)^\lambda}$$

$$= \frac{1}{(1-it)^\lambda}$$

$$= (1-it)^{-\lambda} \quad \text{where } |t| < 1$$

$$\boxed{\phi_X(t) = 1 + \lambda it + \frac{\lambda(\lambda+1)}{2!}(it)^2 + \cdots}$$