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GEOMETRIC DISTRIBUTIONS

A random variable 'x' is said to follow Geometric distribution, if it assumes only non-negative values and its probability mass function is given by,

$$P(X = x) = q^{x} p \quad , x = 0, 1, 2, \ldots \text{ and } 0 < p \leq 1$$

where $q = 1 - p$.

Note: Why it is called a geometric distribution?

1. (Putting $x = 0, 1, 2, 3 \ldots$ we get $q^0 p, qp, q^2 p, q^3 p \ldots$ which are the various terms of geometric progression. Hence it is known as Geometric distribution.

2. We can also take the probability mass function as,

$$P(X = x) = q^{x-1} p \quad , x = 1, 2, \ldots \ \& \ 0 < p \leq 1$$

where $q = 1 - p$.

Mean and Variance:

$$\text{Mean} = \mu_1' = E(X)$$

$$= \sum_{x=0}^{\infty} x\, p(x)$$

$$= \sum_{x=0}^{\infty} x \cdot q^{x} p$$

$$= \sum_{x=0}^{\infty} x\, q^{x} p \cdot q \cdot \frac{1}{q} \quad (\times \ \& \ \div \text{ by } q)$$

$$= \sum_{x=0}^{\infty} x\, q^{x-1} \cdot pq$$

$$= pq \sum_{x=0}^{\infty} x\, q^{x-1}$$

$$= pq \left[1 + 2q + 3q^2 + \cdots \right] \qquad (1-x)^{-2} = 1 + 2q + 3q + \cdots$$

$$= pq\,(1-q)^{-2}$$

$$= pq\,(p+q-q)^{-2} \qquad (\because p+q=1)$$

$$= pq \cdot p^{-2}$$

$$= q\,p^{-1}$$

$$\boxed{\text{Mean} = \frac{q}{p}}$$

$$\mu_2' = \sum_{x=0}^{\infty} x^2\, p(x)$$

$$= \sum_{x=0}^{\infty} x^2\, q^x\, p.$$

$$= \sum_{x=0}^{\infty} \{ x(x-1) + x \}\, q^x\, p$$

$$= \sum_{x=2}^{\infty} x(x-1)\, q^x\, p + \sum_{x=0}^{\infty} x\, q^x\, p$$

$$= \sum_{x=2}^{\infty} x(x-1)\, q^x\, p + \frac{q}{p}$$

$$= \sum_{x=2}^{\infty} x(x-1)\, q^2\, q^{x-2}\, p + \frac{q}{p}$$

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$$\mu_2' = 2pq^2 \left[\sum_{x=2}^{\infty} \frac{x(x-1)q^{x-2}}{2} \right] + \frac{q}{p}$$

$$= 2pq^2 \left[\frac{2(2-1)q^{2-2}}{2} + \frac{3(3-1)q^{3-2}}{2} + \frac{4(4-1)q^{4-2}}{2} + \cdots \right] + \frac{q}{p}$$

$$= 2pq^2 \left[1 + 3q + 6q^2 + \cdots \right] + \frac{q}{p}$$

$$= 2pq^2 (1-q)^{-3} + \frac{q}{p} \qquad \left[\because (1-x)^{-3} = 1 + 3x + 6x^2 + \cdots \right]$$

$$= 2pq^2 (p+q-q)^{-3} + \frac{q}{p}$$

$$= 2pq^2 p^{-3} + \frac{q}{p}$$

$$\mu_2' = \frac{2q^2}{p^2} + \frac{q}{p}$$

$$\text{Variance} = \mu_2' - \mu_1'^2$$

$$= \frac{2q^2}{p^2} + \frac{q}{p} - \left(\frac{q}{p}\right)^2$$

$$= \frac{2q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2}$$

$$= \frac{q^2}{p^2} + \frac{q}{p}$$

$$= \frac{q^2 + pq}{p^2} = \frac{q(p+q)}{p^2} = \frac{q}{p^2}$$

$$\boxed{\text{Variance} = \frac{q}{p^2}}$$

Moment Generating function :

$$M_x(t) = E(e^{tx})$$

$$= \sum_{x=0}^{\infty} e^{tx} P(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} q^x p$$

$$= p \sum_{x=0}^{\infty} e^{tx} q^x$$

$$= p\left[1 + qe^t + (qe^t)^2 + \cdots\right]$$

$$= p(1 - qe^t)^{-1} \qquad \left[\because (1-x)^{-1} = 1 + x + x^2 + \cdots\right]$$

$$\boxed{M_x(t) = \frac{p}{1 - qe^t}}$$

Mean

$$M_x'(t) = p(1 - qe^t)^{-1}$$

$$= -p(1 - qe^t)^{-2}(-qe^t)$$

$$= pq(1 - qe^t)^{-2} e^t$$

$$M_x'(0) = pq(1-q)^{-2} \qquad \frac{pq}{p^2}$$

$$= \frac{q}{p}$$

$$M_x''(t) = pqe^t(-2)(1 - qe^t)^{-3}(-qe^t) + pq(1 - qe^t)^{-2} e^t$$

$$M_x''(0) = \frac{2pq^2}{p^3} + \frac{pq}{p^2}$$

$$= \frac{2q^2}{p^2} + \frac{q}{p}$$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= \frac{2q^2}{p^2} + \frac{q}{p} - \left(\frac{q}{p}\right)^2$$

$$= \frac{2q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2}$$

$$= \frac{q^2}{p^2} + \frac{q}{p}$$

$$= \frac{q^2 + pq}{p^2}$$

$$= \frac{q(p+q)}{p^2} = \frac{q}{p^2}$$

Characteristic function :

$$\varphi_x(t) = E(e^{itx})$$

$$= \sum_{x=0}^{\infty} e^{itx} P(x)$$

$$= \sum_{x=0}^{\infty} e^{itx} q^x p$$

$$= p \sum_{x=0}^{\infty} e^{itx} q^x$$

$$= p\left[1 + qe^{it} + (qe^{it})^2 + \cdots\right]$$

$$= p(1 - qe^{it})^{-1}$$

$$\boxed{\varphi_x(t) = \frac{p}{1 - qe^{it}}}$$

Memoryless property of the Geometric Distribution:

If x is geometrically distributed, then for any two positive integers 's' and 't',

$$P[x > s+t / x > s] = P[x > t]$$

Proof:

The p.m.f of x is,

$$P(x=x) = p(x) = pq^x$$

Consider
$$P(x > s+t) = \sum_{x=s+t}^{\infty} pq^x$$
$$= p[q^{s+t} + q^{s+t+1} + q^{s+t+2} + \dots \infty]$$
$$= pq^{s+t}[1 + q + q^2 + \dots \infty]$$
$$= pq^{s+t}(1-q)^{-1}$$
$$P(x > s+t) = \frac{pq^{s+t}}{1-q}$$

|||ly
$$P(x > s) = \frac{pq^s}{1-q}$$
&
$$P(x > t) = \frac{pq^t}{1-q} \longrightarrow (1)$$

Hence

$$\therefore P[x > s+t / x > s] = \frac{P[x > s+t \cap x > s]}{P(x > s)}$$
$$= \frac{P(x > s+t)}{P(x > s)}$$
$$= \frac{\cancel{p}q^{s+t}/\cancel{1-q}}{\cancel{p}q^s/\cancel{1-q}}$$
$$= q^t$$
$$= \frac{p}{1-q} q^t \quad \left[\because \frac{p}{1-q} = \frac{p}{p} = 1\right]$$

$$P[X > s+t / X > s] = \frac{pq^t}{1-q}$$

$$P[X > s+t / X > s] = P(X > t) \quad (\text{from } (1))$$

Hence geometric distribution lacks memory.

Problems :

(1) If the probability is 0.05 that a certain kind of measuring device will show excessive drift, what is the Probability that the sixth of these measuring devices tes will be the first to show excessive drift ?

Solution :

Let 'x' be the random variable denoting the number of measuring devices to show excessive drift.

Let $p = 0.05$

$q = 1-p = 1-0.05 = 0.95$

$x = 6$

We know that,

$$P(X = x) = p.q^x$$

$$P(X = 6) = (0.05)(0.95)^6$$

$$= \underline{0.0368}$$

(2) If X is a Geometric variate taking values $1, 2, \ldots \infty$, find P(X is odd).

Solution :

We know that,

$$P(X = x) = q^x p \quad , x = 1, 2, \ldots$$

$$\therefore \quad P(X = \text{odd}) = P(X = 1, 3, 5, \ldots)$$

$$= P(X = 1) + P(X = 3) + P(X = 5) + \ldots$$

$$= pq + pq^3 + pq^5 + \ldots$$