



DEPARTMENT OF MATHEMATICS

Given : $f(x,y) = \frac{1}{3}(x+y)$, $0 \leq x \leq 1$, $0 \leq y \leq 2$.

(i) To find Correlation Coefficient:

$$r(x,y) = \frac{\text{Cov}(x,y)}{\sigma_x \sigma_y} \rightarrow \textcircled{1}$$

First let us find $E(x)$, $E(y)$, $E(xy)$, $\text{Var}(x)$ and $\text{Var}(y)$.

$$\begin{aligned} E(x) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x,y) dx dy \\ &= \int_0^2 \int_0^1 x \cdot \frac{1}{3}(x+y) dx dy \\ &= \frac{1}{3} \int_0^2 \int_0^1 (x^2 + xy) dx dy \\ &= \frac{1}{3} \int_0^2 \left[\frac{x^3}{3} + \frac{x^2 y}{2} \right]_0^1 dy \\ &= \frac{1}{3} \int_0^2 \left[\frac{1}{3} + \frac{y}{2} \right] dy \\ &= \frac{1}{3} \left[\frac{y}{3} + \frac{y^2}{4} \right]_0^2 \\ &= \frac{1}{3} \left[\frac{2}{3} + \frac{4}{4} \right] = \frac{1}{3} \left[\frac{2}{3} + 1 \right] = \frac{2+3}{9} \end{aligned}$$

$$E(x) = \frac{5}{9}$$

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$$\begin{aligned} E(Y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f(x,y) dx dy \\ &= \int_0^2 \int_0^1 y \cdot \frac{1}{3} (x+y) dx dy \\ &= \frac{1}{3} \int_0^2 \int_0^1 [xy + \frac{y^2}{1+1}] dx dy \\ &= \frac{1}{3} \int_0^2 \left[\frac{x^2 y}{2} + y^2 x \right]_0^1 dy \\ &= \frac{1}{3} \int_0^2 \left[\frac{xy}{2} + y^2 \right] dy \\ &= \frac{1}{3} \left[\frac{y^2}{4} + \frac{y^3}{3} \right]_0^2 = \frac{1}{3} \left[\frac{4}{4} + \frac{8}{3} \right] \\ &= \frac{3+8}{9} = \frac{11}{9} \end{aligned}$$

$$E(Y) = \frac{11}{9}$$

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 f(x,y) dx dy \\ &= \int_0^2 \int_0^1 x^2 \cdot \frac{1}{3} (x+y) dx dy \\ &= \frac{1}{3} \int_0^2 \int_0^1 (x^3 + x^2 y) dx dy \\ &= \frac{1}{3} \int_0^2 \left[\frac{x^4}{4} + \frac{x^3 y}{3} \right]_0^1 dy \end{aligned}$$

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(40)

$$\begin{aligned} E(x^2) &= \frac{1}{3} \int_0^2 \left[\frac{1}{4} + \frac{y}{3} \right] dy \\ &= \frac{1}{3} \left[\frac{y}{4} + \frac{y^2}{6} \right]_0^2 \\ &= \frac{1}{3} \left[\frac{2}{4} + \frac{4}{6} \right] = \frac{1}{3} \left[\frac{1}{2} + \frac{2}{3} \right] \\ &= \frac{1}{3} \left[\frac{3+4}{6} \right] = \frac{7}{18} \end{aligned}$$

$$E(x^2) = \frac{7}{18}$$

$$\begin{aligned} E(y^2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y^2 f(x,y) dx dy \\ &= \int_0^2 \int_0^1 y^2 \frac{1}{3} (x+y) dx dy \\ &= \frac{1}{3} \int_0^2 \int_0^1 (y^2 x + y^3) dx dy \\ &= \frac{1}{3} \int_0^2 \left[\frac{y^2 x^2}{2} + y^3 x \right]_0^1 dy \\ &= \frac{1}{3} \int_0^2 \left[\frac{y^2}{2} + y^3 \right] dy \\ &= \frac{1}{3} \left[\frac{y^3}{6} + \frac{y^4}{4} \right]_0^2 = \frac{1}{3} \left[\frac{8}{6} + \frac{16}{4} \right] \\ &= \frac{1}{3} \left[\frac{4}{3} + 4 \right] = \frac{1}{3} \left[\frac{4+12}{3} \right] = \frac{16}{9} \end{aligned}$$

$$E(y^2) = \frac{16}{9}$$

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$$\begin{aligned}\text{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= \frac{7}{18} - \left(\frac{5}{9}\right)^2 = \frac{7}{18} - \frac{25}{81}\end{aligned}$$

$$\boxed{\text{Var}(X) = \frac{13}{162}} \Rightarrow \sigma_x^2 = \frac{13}{162} \Rightarrow \sigma_x = \sqrt{\frac{13}{162}}$$

$$\begin{aligned}\text{Var}(Y) &= E(Y^2) - [E(Y)]^2 \\ &= \frac{16}{9} - \left(\frac{11}{9}\right)^2\end{aligned}$$

$$\boxed{\text{Var}(Y) = \frac{23}{81}} \Rightarrow \sigma_y^2 = \frac{23}{81} \Rightarrow \sigma_y = \sqrt{\frac{23}{81}}$$

$$E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f(x,y) dx dy$$

$$= \int_0^2 \int_0^1 xy \frac{1}{3} (x+y) dx dy$$

$$= \frac{1}{3} \int_0^2 \int_0^1 (x^2y + xy^2) dx dy$$

$$= \frac{1}{3} \int_0^2 \left[\frac{x^3y}{3} + \frac{x^2y^2}{2} \right]_0^1 dy$$

$$= \frac{1}{3} \int_0^2 \left[\frac{y}{3} + \frac{y^2}{2} \right] dy$$

$$= \frac{1}{3} \left[\frac{y^2}{6} + \frac{y^3}{6} \right]_0^2 = \frac{1}{18} (4 + 8) = \frac{12}{18}$$

$$\boxed{E(XY) = \frac{2}{3}}$$

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$$\begin{aligned} \text{Cov}(X,Y) &= E(XY) - E(X)E(Y) \\ &= \frac{2}{3} - \frac{5}{9} \cdot \frac{11}{9} \\ &= \frac{2}{3} - \frac{55}{81} = \frac{-1}{81} \end{aligned}$$

$$\boxed{\text{Cov}(X,Y) = \frac{-1}{81}}$$

$$\begin{aligned} \therefore r(X,Y) &= \frac{\text{Cov}(X,Y)}{\sigma_x \sigma_y} \\ &= \frac{-1/81}{\sqrt{\frac{13}{162} \times \frac{23}{81}}} = -\sqrt{\frac{2}{299}} \end{aligned}$$

$$\boxed{r = -\sqrt{\frac{2}{299}} = -0.0818}$$

(ii) Two lines of regression:

Equation of the line of regression of X on Y is,

$$x - \bar{x} = r \cdot \frac{\sigma_x}{\sigma_y} (y - \bar{y})$$

$$x - \frac{5}{9} = (-0.0818) \frac{\sqrt{13/162}}{\sqrt{23/81}} (y - \frac{11}{9})$$

$$\boxed{x - \frac{5}{9} = (-0.0435) (y - \frac{11}{9})}$$

Equation of the line of regression of Y on X is,

$$y - \bar{y} = r \cdot \frac{\sigma_y}{\sigma_x} (x - \bar{x})$$

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