DIVIDED DIFFERENCES :

Let $f(x_0), f(x_1) \ldots f(x_n)$ be the values of the function $y = f(x)$ corresponding to $x = x_0, x_1, \ldots x_n$ which are not neccessarily equally spaced.

I order divided difference of $f(x)$ for the arguments x_0 and x_1 is

$$f(x_0, x_1) = \underset{x_1}{\Delta} f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f(x_1, x_2) = \underset{x_2}{\Delta} f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$f(x_2, x_3) = \underset{x_3}{\Delta} f(x_2) = \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$ and so on

In general, $f(x_{n-1}, x_n) = \underset{x_n}{\Delta} f(x_{n-1}) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$

II order divided difference of $f(x)$ for x_0, x_1, x_2 is

$$f(x_0,x_1,x_2) = \underset{x_1,x_2}{\Delta^2} f(x_0) = \frac{f(x_1,x_2) - f(x_0,x_1)}{x_2 - x_0}$$

$$f(x_1,x_2,x_3) = \underset{x_2,x_3}{\Delta^2} f(x_1) = \frac{f(x_2,x_3) - f(x_1,x_2)}{x_3 - x_1}$$

III order divided difference of $f(x)$ for x_0, x_1, x_2, x_3 is

$$f(x_0,x_1,x_2,x_3) = \underset{x_1,x_2,x_3}{\Delta^3} f(x_0) = \frac{f(x_1,x_2,x_3) - f(x_0,x_1,x_2)}{x_3 - x_0}$$

n^{th} order divided difference:

$$f(x_0,x_1,x_2,\ldots,x_n) = \underset{x_1,x_2\ldots x_n}{\Delta^n} f(x_0) = \frac{f(x_1,x_2,\ldots x_n) - f(x_0,x_1,\ldots x_{n-1})}{x_n - x_0}$$

Divided difference table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x_0	$f(x_0)$			
		$f(x_0,x_1)$		
x_1	$f(x_1)$		$f(x_0,x_1,x_2)$	
		$f(x_1,x_2)$		$f(x_0,x_1,x_2,x_3)$
x_2	$f(x_2)$		$f(x_1,x_2,x_3)$	
		$f(x_2,x_3)$		$f(x_1,x_2,x_3,x_4)$
x_3	$f(x_3)$		$f(x_2,x_3,x_4)$	
		$f(x_3,x_4)$		
x_4	$f(x_4)$			

1) Form the divided difference table for the following data:

x:	1	2	4	7	12
$f(x)$:	22	30	82	106	206

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	22				
		$\frac{30-22}{2-1}=8$			
2	30		$\frac{26-8}{4-1}=6$		
		$\frac{82-30}{4-2}=26$		$\frac{-3.6-6}{7-1}=-1.6$	
4	82		$\frac{8-26}{7-2}=-3.6$		$\frac{0.51+1.6}{12-1}=0.19$
		$\frac{106-82}{7-4}=8$		$\frac{1.5+3.6}{12-2}=0.51$	
7	106		$\frac{20-8}{12-4}=1.5$		
		$\frac{206-106}{12-7}=20$			
12	206				

2) Form the divided difference table:

x :	2	4	5	7	8
$f(x)$:	3	43	138	778	1515

PROPERTIES :

(1) The divided differences are symmetrical in all their arguments i.e., independent of the order of arguments

(2) The divided difference operator Δ is linear.

(3) The n^{th} divided differences of a polynomial of the n^{th} degree are constant.

1) S.T. $\underset{bcd}{\Delta^3}\left(\frac{1}{a}\right) = -\frac{1}{abcd}$

Soln: If $f(x) = \frac{1}{x} \Rightarrow f(a) = \frac{1}{a}$; $f(b) = \frac{1}{b}$

$$f(a,b) = \underset{b}{\Delta} f(a) = \frac{f(b)-f(a)}{b-a}.$$

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$$= \underset{b}{\Delta}\left(\frac{1}{a}\right) = \frac{\frac{1}{b}-\frac{1}{a}}{b-a} = -\frac{1}{ab}$$

$$f(a,b,c) = \underset{b,c}{\Delta^2} f(a) = \frac{f(b,c)-f(a,b)}{c-a}$$

$$= \underset{b,c}{\Delta^2}\left(\frac{1}{a}\right) = \frac{-\frac{1}{bc}+\frac{1}{ab}}{c-a} = \frac{1}{abc}$$

$$f(a,b,c,d) = \underset{bcd}{\Delta^3}\left(\frac{1}{a}\right) = \frac{\frac{1}{bcd}-\frac{1}{abc}}{d-a} = -\frac{1}{abcd}$$

$$\Rightarrow \underset{bcd}{\Delta^3}\left(\frac{1}{a}\right) = -\frac{1}{abcd}.$$

NEWTON'S DIVIDED DIFFERENCE FORMULA (or)

NEWTON'S INTERPOLATION FORMULA [UNEQUAL INTER

Let $y = f(x)$ takes values $f(x_0), f(x_1), \ldots f(x_n)$ corresponding to the arguments $x_0, x_1, \ldots x_n$ then Newton's Interpolation formula is

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) + \ldots + (x-x_0)(x-x_1)\cdots(x-x_{n-1}) f(x_0, x_1, \ldots x_n)$$

) using Newton's divided difference formula determine $f(3)$ from the data :

x :	0	1	2	4	5
$f(x)$:	1	14	15	5	6

Soln:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	1				
		$\frac{14-1}{1-0}=13$			
1	14		$\frac{1-13}{2-0}=-6$		
		$\frac{15-14}{2-1}=1$		$\frac{-2+6}{4-0}=1$	
2	15				$\frac{1-1}{5-0}=0$
			$\frac{-5-1}{4-1}=-2$		
4	5	$\frac{5-15}{4-2}=-5$		$\frac{2+2}{5-1}=1$	
			$\frac{1+5}{5-2}=2$		
5	6	$\frac{6-5}{5-4}=1$			

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2)$$
$$+ (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3) +$$
$$(x-x_0)(x-x_1)(x-x_2)(x-x_3) f(x_0,x_1,x_2,x_3,x_4)$$

$x_0 = 0$; $x_1 = 1$; $x_2 = 2$; $x_3 = 4$; $x_4 = 5$

$f(x_0) = 1$; $f(x_0,x_1) = 13$; $f(x_0,x_1,x_2) = -6$;

$f(x_0,x_1,x_2,x_3) = 1$; $f(x_0,x_1,x_2,x_3,x_4) = 0$

(2) Using Newton's divided difference formula find the missing value from the table:

x:	1	2	4	5	6
y:	14	15	5	–	9

Soln: $f(5) = 3$

(3) Using Newton's divided difference formula, find $u(3)$ given $u(1) = -26$, $u(2) = 12$, $u(4) = 256$, $u(6) = 844$

Soln: $u(3) = 100$.