$$f_U(u) = \int_{-\infty}^{\infty} f_{UV}(u,v)\,dv$$

$$= \int_{u}^{\infty} e^{u-2v}\,dv$$

$$= e^{u}\int_{u}^{\infty} e^{-2v}\,dv$$

$$= e^{u}\left[\frac{e^{-2v}}{-2}\right]_{u}^{\infty}$$

$$= \frac{e^{u}}{2}\left[0+e^{-2u}\right]$$

$$f_U(u) = \frac{e^{-u}}{2}$$

Step 6 :

$$\therefore f_U(u) = \begin{cases} e^{-u/2}, & 0<v<\infty \cdot u>0 \\ e^{u/2}, & u<0 \end{cases}$$

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(4) If X and Y are independent random variables with pdf's $e^{-x}, x>0$ and $e^{-y}, y>0$ respectively, find the density function of $U = \frac{X}{X+Y}$ and $V = X+Y$. Are U & V independent.

$$= e^{-(x+y)}$$

Solution :

Step 1 :

Given : $f(x) = e^{-x}, \; x > 0$

$f(y) = e^{-y}, \; y > 0$

Since X and Y are independent,

$$f(x,y) = f(x) \cdot f(y)$$

$$= e^{-x} \cdot e^{-y}$$

$$f(x,y) = e^{-(x+y)}, \quad x>0, y>0$$

Step 2 :

Let $U = \dfrac{X}{X+Y}$

$V = X + Y$

i.e., $u = \dfrac{x}{x+y}$ , $v = x+y$

$y = v - x$

$\therefore \cancel{u = \dfrac{x}{x+v-x}}$

$u(x+y) = x$

$uv = x$

$x = uv$

$\dfrac{\partial x}{\partial u} = v \; ; \; \dfrac{\partial x}{\partial v} = u$

$y = v - uv$

$y = v(1-u)$

$\dfrac{\partial y}{\partial v} = 1-u$

$\dfrac{\partial y}{\partial u} = -v$

$$|J| = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = |v(1-u)+vu|$$

$$= v$$

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**Step 3 :**

$$f_{UV}(u,v) = f_{XY}(x,y)\,|J|$$
$$= e^{-(x+y)} \cdot v$$
$$= e^{-(uv+v-uv)} \cdot v$$
$$\boxed{f_{UV}(u,v) = v e^{-v}}$$

**Step 4 :**

Given : $x > 0$ & $y > 0$

$uv > 0$ & $v - uv > 0$

$v > 0$ & $v > uv$

$1 > u$

$0 < u < 1$

**Step 5 :**

Hence the pdf is given by,

$$f_{UV}(u,v) = ve^{-v}, \quad 0<u<1, \ v>0$$

**Step 6 :**

The density function of $U$ is,

$$f_U(u) = \int_{-\infty}^{\infty} f_{UV}(u,v)\, dv$$
$$= \int_{0}^{\infty} ve^{-v}\, dv$$

$u = v$ , $v = e^{-v}$

$u' = 1$ , $v_1 = -e^{-v}$

$v_2 = e^{-v}$

$$= \left[-ve^{-v} - e^{-v}\right]_0^{\infty}$$
$$\boxed{f_U(u) = 1 \quad \text{for } 0<u<1}$$

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Solution:

The density function of v is,

$$f_V(v) = \int_{-\infty}^{\infty} f_{UV}(u,v)\,du$$

$$= \int_0^1 v e^{-v}\,du$$

$$= v e^{-v}[u]_0^1$$

$$\boxed{f_V(v) = v e^{-v}, \quad v > 0}$$

Step 7: To test U & V are independent:

$$f_{UV}(u,v) = f_U(u) \cdot f_V(v)$$

$$= 1 \cdot v e^{-v}$$

$$= v e^{-v}$$

$$\therefore f_{UV}(u,v) = f_U(u) \cdot f_V(v)$$

Hence the random variables U & V are independent