DISCRETE DISTRIBUTIONS :

The important discrete distributions of a random variable 'x' are,

1. Binomial distribution
2. Poisson distribution
3. Geometric distribution
4. Negative Binomial distribution.

BINOMIAL DISTRIBUTION :

The probability mass function of a random variable 'x' which follows the binomial distribution is,

$$P(X=x) = nC_x \, p^x \, q^{n-x} \quad , \quad x = 0,1,2,\dots n \ \& \ p+q=1$$

$$(q+p)^n = q^n + nC_1 \, q^{n-1} p + nC_2 \, q^{n-2} p^2 + \dots + nC_x \, p^x q^{n-x}$$

which is a binomial series and hence the distribution is called a Binomial distribution.

NOTE :

$$P(0 \text{ success}) = nC_0 \, p^0 \, q^{n-0} = q^n$$

$$P(1 \text{ success}) = nC_1 \, p^1 q^{n-1}$$

$$P(2 \text{ success}) = nC_2 \, p^2 q^{n-2} \quad \text{and so on.}$$

ASSUMPTIONS :

(i) There are only two possible outcomes for each trial (success or failure)
(ii) The probability of a success is the same for each trial
(iii) There are 'n' trials, where 'n' is a constant.
(iv) The 'n' trials are independent.

Moment Generating function (M.G.F) :

$$M_x(t) = E(e^{tx})$$

$$= \sum_{x=0}^{n} e^{tx} p(x)$$

$$= \sum_{x=0}^{n} e^{tx} \, nc_x \, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} (pe^t)^x \, nc_x \, q^{n-x}$$

$$= q^n + nc_1 . (pe^t) q^{n-1} + nc_2 (pe^t)^2 q^{n-2} + \ldots\ldots$$

$$\boxed{M_x(t) = (q+pe^t)^n}$$

$$(x+a)^n = a^n + nc_1 x a^{n-1} + nc_2 x^2 a^{n-2} + \ldots + nc_n x^n a$$

Mean and Variance :

$$M_x(t) = (q+pe^t)^n$$

$$M_x'(t) = n(q+pe^t)^{n-1} pe^t$$

$$M_x'(0) = n(q+p)^{n-1} p$$

$$M_x'(0) = np \qquad (\because p+q=1)$$

$$\therefore \boxed{\text{Mean} = E(x) = np}$$

$$M_x''(t) = np\left[(q+pe^t)^{n-1} \cdot e^t + e^t (n-1)(q+pe^t)^{n-2} pe^t\right]$$

Putting $t=0$,

$$M_x''(0) = np\left[(q+p)^{n-1} + (n-1)(q+p)^{n-2} p\right]$$

$$= np\left[1+(n-1)p\right]$$

$$= np + n^2p^2 - np^2$$

$$M_x''(0) = n^2p^2 + np(1-p)$$

$$\boxed{M_x''(0) = E(X^2) = n^2p^2 + npq}$$

$$\therefore \text{Variance} = E(X^2) - [E(X)]^2$$

$$= n^2p^2 + npq - (np)^2$$

$$= \cancel{n^2p^2} + npq - \cancel{n^2p^2}$$

$$\boxed{\text{Variance} = npq}$$

$$\text{Standard deviation} = \sqrt{\text{Variance}}$$

$$\boxed{SD = \sqrt{npq}}$$

PROBLEMS :

(1) The mean and SD of a binomial distribution are 20 and 4. Determine the distribution.

Solution :

$$\text{Mean} = np = 20 \rightarrow (1)$$

$$SD = \sqrt{npq} = 4$$

$$npq = 4^2 = 16 \rightarrow (2)$$

$$\frac{(2)}{(1)} = \frac{n\cancel{p}q}{n\cancel{p}} = \frac{16}{20} \Rightarrow \boxed{q = \frac{4}{5}}$$

$$p = 1 - q = 1 - \frac{4}{5} = \frac{1}{5}.$$

$$\boxed{p = \frac{1}{5}}$$

subs p in (1),

$$n \cdot \frac{1}{5} = 20$$

$$\boxed{n = 100}$$

$\therefore$ The binomial distribution is,

$$P(X = x) = p(x) = nC_x \, p^x q^{n-x}$$

$$= 100C_x \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{100-x}, x = 0, 1, \dots 100$$

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(2) If X is a binomial random variable with expected value 2 and variance 4/3, find $P(X=5)$.

Solution :

$$E(X) = 2$$

$$np = 2 \rightarrow (1)$$

$$\text{variance} = 4/3$$

$$npq = 4/3 \rightarrow (2)$$

$$\frac{(2)}{(1)} = \frac{npq}{np} = \frac{4/3}{2} = \frac{4}{3} \times \frac{1}{2} = \frac{2}{3}$$

$$\boxed{q = \frac{2}{3}}$$

$$p = 1 - q = 1 - \frac{2}{3} = \frac{1}{3} \Rightarrow \boxed{p = \frac{1}{3}}$$

subs p in (1),

$$n \cdot \frac{1}{3} = 2 \Rightarrow \boxed{n = 6}$$

$$P(X = x) = nC_x \, p^x \, q^{n-x}$$

$$P(X = 5) = 6C_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^{6-5}$$

$$= 6C_1 \, \frac{1}{3^5} \cdot \frac{2}{3} = \frac{\cancel{6}^{\,2} \times 2}{\cancel{3} \times 3 \times 3 \times 3 \times 3 \times 3}$$

$$\boxed{P(X = 5) = 0.0165}$$

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(3) If the MGF of a r.v X is of the form $(0.4e^t + 0.6)^8$ What is the MGF of $3X+2$. Find $E(X)$, $Var(X)$, $P(X=2)$.

Solution :

$$M_X(t) = (0.4e^t + 0.6)^8 = E(e^{tX})$$

$$\text{M.G.F of } 3X+2 = E\left(e^{t(3X+2)}\right)$$

$$= E(e^{2t} \cdot e^{3tX})$$

$$= e^{2t} E(e^{3tX})$$

$$= e^{2t} E(e^{X(3t)})$$

$$\boxed{M_{3X+2}(t) = e^{2t}(0.4e^{3t} + 0.6)^8}$$

$$M_X(t) = (0.4e^t + 0.6)^8$$

But $M_X(t) = (q + pe^t)^n$

$\therefore \quad q = 0.6, \; p = 0.4 \;\&\; n = 8$

$E(X) = np = 8 \times 0.4 = 3.2$

$\boxed{E(X) = 3.2}$

$\text{Variance} = E(X^2) - npq = 8 \times 0.4 \times 0.6$

$\boxed{\text{Variance} = 1.92}$

$P(X = x) = nC_x \, p^x \, q^{n-x}$

$P(X = 2) = 8C_2 \, (0.4)^2 \, (0.6)^{8-2}$

$= \frac{8 \times 7}{2} \times 0.16 \times 0.0467$

$\boxed{P(X = 2) = 0.2092}$

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(4) If 10 % of the screws produced by an automatic machine are defective, find the probability that out of 20 screws selected at random, there are (i) exactly 2 defective (ii) atmost 3 defective (iii) atleast 2 defectives and (iv) between 1 and 3 defectives (inclusive).

Solution :

Let X be the R.V denoting the number of defective screws.

$p = 10\% = \frac{10}{100} = \frac{1}{10}$

$q = 1 - p = 1 - \frac{1}{10} = \frac{9}{10}$

$n = 20$

$P(X = x) = nC_x \, p^x \, q^{n-x}$

(i) P(getting exactly 2 defectives)

$= P(X = 2)$

$= 20C_2 \left(\frac{1}{10}\right)^2 \left(\frac{9}{10}\right)^{20-2}$

$= \frac{20 \times 19}{2} \times \frac{1}{100} \times \left(\frac{9}{10}\right)^{18}$

P(getting exactly 2 defectives) = $\underline{0.2852}$

(ii) P(getting atmost 3 defective) = $P(X \leq 3)$

$$= P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= 20C_0 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^{20-0} + 20C_1 \left(\frac{1}{10}\right)^1 \left(\frac{9}{10}\right)^{20-1} + 20C_2 \left(\frac{1}{10}\right)^2 \left(\frac{9}{10}\right)^{20-2} + 20C_3 \left(\frac{1}{10}\right)^3 \left(\frac{9}{10}\right)^{20-3}$$

$$= \left(\frac{9}{10}\right)^{20} + \frac{20}{10}\left(\frac{9}{10}\right)^{19} + \frac{20 \times 19}{2} \times \frac{1}{100}\left(\frac{9}{10}\right)^{18} + \frac{20 \times 19 \times 18}{2 \times 3} \times \frac{1}{1000}\left(\frac{9}{10}\right.$$

$$= 0.1216 + 0.2702 + 0.2852 + 0.1901$$

$$= \underline{0.8671}$$

(iii) P(getting atleast 2 defective = $P(X \geq 2)$

$$= 1 - P(X < 2)$$

$$= 1 - [P(X=0) + P(X=1) + P(X=2)]$$

$$= 1 - [0.1216 + 0.2702 + 0.2852]$$

$$= \underline{0.323}$$

(iv) P(getting number of defectives between 1 and 3) (including)

$$= P(1 \leq X \leq 3)$$

$$= P(X=1) + P(X=2) + P(X=3)$$

$$= 0.2702 + 0.2852 + 0.1901$$

$$= \underline{0.7455}$$