SNS COLLEGE OF TECHNOLOGY

Coimbatore-35
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DEPARTMENT OF ELECTRONICS & COMMUNICATION ENGINEERING

19ECT212 – CONTROL SYSTEMS

II YEAR/ IV SEMESTER

UNIT I – CONTROL SYSTEM MODELING

TOPIC 3- DIFFERENTIAL EQUATION

OUTLINE

- REVIEW ABUT PREVIOUS CLASS
- WHAT IS A DIFFERENTIAL EQUATION & TYPES
- **TYPES OF ODE**
- **FIRST ORDER ODE-FIRST ORDER LINEAR ODE**
- **BERNOULI EQUATIONS**
- **SECOND ORDER ODE**
- **ACTIVITY**
- INITIAL / BOUNDARY VALUE PROBLEMS
- **Higher Order Homogeneous Differential ODE**
- **APPLICATIONS OF ODE**
- **EXAMPLES OF PDE**
  - Laplace Equation
  - Heat Equation
  - Wave Equation

SUMMARY

WHAT IS DIFFERENTIAL EQUATION? & TYPES

A Differential Equation is an equation containing the derivative of one or more dependent variables with respect to one or more independent variables.

1. Ordinary Differential Equations.
2. Partial Differential Equations.
3. Linear Differential Equations.
4. Non-linear differential equations.
5. Homogeneous Differential Equations.
6. Non-homogenous Differential Equations

For Example,

$$\frac{dy}{dx} = 2xy \qquad x\frac{dy}{dx} = y - 1 \qquad \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \frac{\partial x}{\partial y} + \frac{\partial y}{\partial x} + 2 = 0 \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 2\frac{\partial u}{\partial t}$$

TYPES OF DIFFERENTIAL EQUATION

ODE (ORDINARY DIFFERENTIAL EQUATION):

An equation contains only ordinary derivates of one or more dependent variables of a single independent variable.

For Example,

$dy/dx + 5y = e^x$, $(dx/dt) + (dy/dt) = 2x + y$

PDE (PARTIAL DIFFERENTIAL EQUATION):

An equation contains partial derivates of one or more dependent variables of two or more independent variables.

For Example,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 2\frac{\partial u}{\partial t} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

TYPES OF ODE

- **FIRST ORDER ODE**
  - FIRST ORDER LINEAR ODE
  - EXACT EQUATION
  - NON-LINEAR FIRST ORDER ODE
  - SEPERABLE EQUATION
  - BERNOULLI DIFFERENTIAL EQUATION
- **SECOND ORDER ODE**
  - LINEAR SECOND ORDER ODE
  - HOMOGENEOUS SECOND ORDER ODE
  - INITIAL AND BOUNDARY VALUE PROBLEMS
  - NON-LINEAR SECOND ORDER ODE
  - NON-HOMOGENEOUS SECOND ORDER ODE
- **HIGHER ORDER ODE**
  - LINEAR NTH ORDER ODE
  - HOMOGENEOUS EQUATION
  - NON-HOMOGENEOUS EQUATION

FIRST ORDER ODE

- A first order differential equation is an equation involving the unknown function y, its derivative y' and the variable x. We will only talk about explicit differential equations.

$$y' = f(x, y)$$

- General Form,

$$\frac{dy}{dx} = F(x, y),$$

- For Example,

$$\frac{dy}{dx} = 2x + 3$$

FIRST ORDER LINEAR ODE

A **first order linear differential equation** has the following form:

$$\frac{dy}{dx} + p(x)y = q(x).$$

The general solution is given by

$$y = \frac{\int u(x)q(x)dx + C}{u(x)},$$

Where

$$u(x) = \exp\left(\int p(x)dx\right).$$

called the **integrating factor**. If an initial condition is given, use it to find the constant *C*.

EXACT EQUATION

- Let a first order ordinary differential equation be expressible in this form:
  $M(x,y)+N(x,y)dy/dx=0$
  such that M and N are *not* homogeneous functions of the same degree.

- However, suppose there happens to exist a function f(x,y) such that:

  $\partial f/\partial x=M, \quad \partial f/\partial y=N$
  such that the second partial derivatives of f exist and are continuous.

- Then the expression Mdx+Ndy is called an **exact differential**, and the differential equation is called an **exact differential equation**.

SEPARABLE DIFFERENTIAL EQUATIONS

- A separable differential equation is any differential equation that we can write in the following form.

$$N(y)\frac{dy}{dx} = M(x)$$

- Note that in order for a differential equation to be separable all the *y*'s in the differential equation must be multiplied by the derivative and all the *x*'s in the differential equation must be on the other side of the equal sign.

$$N(y)\,dy = M(x)\,dx$$

BERNOULI EQUATIONS

$$y' + p(x)y = q(x)y^n$$

- where *p(x)* and *q(x)* are continuous functions on the interval we're working

on and *n* is a real number. Differential equations in this form are called **Bernoulli Equations**.

- If or then the equation is linear and we already know how to solve it in these cases. Therefore, in this section we're going to be looking at solutions for values of *n* other than these two.
- In order to solve these we'll first divide the differential equation by y^n to get,
- We are now going to use the substitution to convert this into a differential equation in terms of *v*. As we'll see this will lead to a differential equation that we can solve.

$$y^{-n} y' + p(x) y^{1-n} = q(x)$$

$$v' = (1-n) y^{-n} y'$$

SECOND ORDER ODE

The most general linear second order differential equation is in the form.

$$p(t)y'' + q(t)y' + r(t)y = g(t)$$

In fact, we will rarely look at non-constant coefficient linear second order differential equations. In the case where we assume constant coefficients we will use the following differential equation.

$$ay'' + by' + cy = g(t)$$

Initially we will make our life easier by looking at differential equations with $g(t) = 0$.

When $g(t) = 0$ -- **Differential Equation Homogeneous** and when ---**Differential Equation Non- Homogeneous.**

$$g(t) \neq 0$$

SECOND ORDER ODE...ACTIVITY

- how to go about solving a constant coefficient, homogeneous, linear, second order differential equation.
- Here is the general constant coefficient, homogeneous, linear, second order differential equation.

$$ay'' + by' + cy = 0$$

- For Example,

$$y'' - 9y = 0$$

- CONNECTIONS....START WITH THE WORD ......

INITIAL / BOUNDARY VALUE PROBLEMS

- conditions specified at the extremes ("boundaries") of the independent variable in the equation .
- all of the conditions specified at the same value of the independent variable (and that value is at the lower boundary of the domain, thus the term "initial" value).
  - For example, if the independent variable is time over the domain [0,1], a boundary value problem would specify values for at both and , whereas an initial value problem would specify a value of and at time .
- EXAMPLE:
  - Finding the temperature at all points of an iron bar with one end kept at absolute zero and the other end at the freezing point of water would be a boundary value problem.
  - If the problem is dependent on both space and time, one could specify the value of the problem at a given point for all time the data or at a given time for all space.
  - Concretely, an example of a boundary value (in one spatial dimension) is the problem

INITIAL / BOUNDARY VALUE PROBLEMS

- To solve for the unknown function $y(x)$ with the boundary conditions

$$y(0) = 0,\ y(\pi/2) = 2.$$

- Without the boundary conditions, the general solution to this equation is

$$y(x) = A\sin(x) + B\cos(x).$$

- From the boundary condition $y(0) = 0$ one obtains

$$0 = A \cdot 0 + B \cdot 1$$

- which implies that $B = 0.$ From the boundary condition $y(\pi/2) = 2$ one finds

$$2 = A \cdot 1$$

- and so $A = 2.$ One sees that imposing boundary conditions allowed one to determine a unique solution, which in this case is

$$y(x) = 2\sin(x).$$

Higher Order Homogeneous Differential ODE

- For Example,

$$y^{(3)} - 5y'' - 22y' + 56y = 0 \qquad y(0) = 1 \quad y'(0) = -2 \quad y''(0) = -4$$

- The above equation is an example of **Higher Order Homogeneous Differential ODE** with initial conditions.

$$y^{(3)} - 12y'' + 48y' - 64y = 12 - 32e^{-8t} + 2e^{4t}$$

- **Similarly**, the above equation is an **Higher Order Non-Homogeneous Differential ODE** with coefficients.

APPLICATIONS OF ODE

- ❖ **MODELLING WITH FIRST-ORDER EQUATIONS**
  - Newton's Law of Cooling
  - Electrical Circuits
- ❖ **MODELLING FREE MECHANICAL OSCILLATIONS**
  - No Damping
  - Light Damping
  - Heavy Damping
- ❖ **MODELLING FORCED MECHANICAL OSCILLATIONS**
- ❖ **COMPUTER EXERCISE OR ACTIVITY**

LINEAR & NON- LINEAR PDE

A PDEis linear if it is linear in theunknown functionand its derivatives

Example of linear PDE :

$$2\,u_{xx} + 1\,u_{xt} + 3\,u_{tt} + 4\,u_{x} + \cos(2t) = 0$$

$$2\,u_{xx} - 3\,u_{t} + 4\,u_{x} = 0$$

Examples of Nonlinear PDE

$$2\,u \quad + \left(u \quad\right) + 3\,u \quad = 0$$

$$u_{xx} + 2\,u_{xt} + 3u_{t} = 0$$

$$2\,u_{xx} + 2\,u_{xt}\,u_{t} + 3u_{t} = 0$$

EXAMPLES OF PDE

PDEs are used to model many systems in many different fields of science and engineering.

Important Examples:

- Laplace Equation
- Heat Equation
- Wave Equation

LAPLACE EQUATION

- Laplace Equation is used to describe the steady state distribution of heat in a body.
- Also used to describe the steady state distribution of electrical charge in a body.

$$\frac{\partial^2 u(x,y,z)}{\partial x^2} + \frac{\partial^2 u(x,y,z)}{\partial y^2} + \frac{\partial^2 u(x,y,z)}{\partial z^2} = 0$$

HEAT EQUATION

- The function u(x,y,z,t) is used to represent the temperature at time *t* in a physical body at a point with coordinates (x,y,z)
- **α** is the thermal diffusivity. It is sufficient to consider the case **α = 1**.

$$\frac{\partial u(x,y,z,t)}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

WAVE EQUATION

- The function $u(x,y,z,t)$ is used to represent the displacement at time t of a particle whose position at rest is (x,y,z) .
- The constant c represents the propagation speed of the wave.

$$\frac{\partial^2 u(x,y,z,t)}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

SUMMARY