(2) The differential equation of a physical phenomenon is given by

$$\frac{d^2y}{dx^2} + 500x^2 = 0 \; ; \quad 0 \le x \le 1$$

By using the trial function, $y = a_1(x - x^3) + a_2(x - x^5)$, Calculate the value of the parameters a_1 and a_2 by the following methods:
(i) Point collocation method (ii) Subdomain collocation
(iii) Least Squares (iv) Galerkin.
The boundary conditions are: $y(0) = 0$; $y(1) = 0$

Soln:

$$\frac{d^2y}{dx^2} + 500x^2 = 0 \; ; \quad 0 \le x \le 1$$

Trial function $y = a_1(x - x^3) + a_2(x - x^5)$

Boundary conditions are, $y(0) = 0$
$y(1) = 0$

Checking boundary condition on trial function

* $y(0) = 0$.

i.e., when $x = 0$.

$$a_1(0 - 0^3) + a_2(0 - 0^5) = 0$$

* $y(1) = 0$
i.e., when $x = 1$

$$a_1(1 - 1^3) + a_2(1 - 1^5) = 0$$

- Boundary conditions are satisfied.

Residual, R:

$$y = a_1(x - x^3) + a_2(x - x^5)$$

$$\frac{dy}{dx} = a_1(1 - 3x^2) + a_2(1 - 5x^4)$$

$$\frac{d^2y}{dx^2} = a_1(0 - 3 \cdot 2x) + a_2(0 - 5 \cdot 4x^3)$$

$$\frac{d^2y}{dx^2} = -6a_1x - 20a_2x^3$$

Substituting $\frac{d^2y}{dx^2}$ value in given differential eqn.

$$\frac{d^2y}{dx^2} + 500x^2 = 0.$$

$$\boxed{R = -6a_1x - 20a_2x^3 + 500x^2}$$

The interval 0 to 1 is divided into two domains

Domain - 1 = 0 to ½.

Domain - 2 = ½ to 1

(i) Point Collocation method :

* Domain 1 : Limits 0 to ½

put $x = \frac{1}{3}$.

$$R = -\cancel{6}^{2}a_1\left(\frac{1}{\cancel{3}}\right) - 20a_2\left(\frac{1}{3}\right)^3 + 500\left(\frac{1}{3}\right)^2 = 0.$$

$$-2a_1 - \frac{20}{27}a_2 + \frac{500}{9} = 0.$$

$$a_1 + 0.37059a_2 = 27.775 \quad \cdots\cdots> \text{Ⓐ}$$

* Domain 2 : Limits ½ to 1

Put $x = \frac{2}{3}$

$$R = -\cancel{6}^{2}a_1\left(\frac{2}{\cancel{3}}\right) - 20a_2\left(\frac{2}{3}\right)^3 + 500\left(\frac{2}{3}\right)^2 = 0$$

$$-4a_1 - 20a_2\left(\frac{8}{27}\right) + 500\left(\frac{4}{9}\right) = 0$$

$$a_1 + 1.481a_2 = 55.555 \quad \cdots\cdots> \text{Ⓑ}$$

Solving Ⓐ & Ⓑ

$$a_1 = 18.53$$

$$a_2 = 25$$

ii) Subdomain collocation :

$$\int_0^1 R\,dx = 0$$

* Domain 1

$$\int_0^{1/2} R.\,dx = 0.$$

$$\int_0^{1/2} (-6a_1x - 20a_2x^3 + 500x^2)\,dx = 0.$$

$$\left[-6a_1\left[\frac{x^2}{2}\right] - 20a_2\left[\frac{x^4}{4}\right] + 500\left(\frac{x^3}{3}\right)\right]_0^{1/2} = 0.$$

$$a_1 + 0.4166\,a_2 = 27.773 \;\cdots\to \text{C}$$

* Domain 2

$$\int_{1/2}^1 R.\,dx = 0.$$

$$\int_{1/2}^1 (-6a_1x - 20a_2x^3 + 500x^2)\,dx = 0.$$

$$\left(-6a_1\left[\frac{x^2}{2}\right] - 20a_2\left[\frac{x^4}{4}\right] + 500\left[\frac{x^3}{3}\right]\right)_{1/2}^{1} = 0$$

$$a_1 + 2.083a_2 = 64.813 \;\cdots\to \text{D}$$

Solving C & D

$$a_1 = 18.50$$

$$a_2 = 22.23$$

(iii) Least Squares Method:

$$I = \int_0^1 R^2\, dx$$

$$\frac{\partial I}{\partial a_1} = \int_0^1 R \cdot \frac{\partial R}{\partial a_1}\, dx$$

* Domain 1

$$\frac{\partial I}{\partial a_1} = \int_0^{1/2} R \cdot \frac{\partial R}{\partial a_1}\, dx$$

$$R = -6a_1 x - 20a_2 x^3 + 500x^2$$

$$\frac{\partial R}{\partial a_1} = -6x$$

$$\Rightarrow \frac{\partial I}{\partial a_1} = \int_0^{1/2} (-6a_1 x - 20a_2 x^3 + 500x^2)(-6x)\, dx$$

$$\frac{\partial I}{\partial a_1} = 0,$$

$$\Rightarrow \int_0^{1/2} (-6a_1 x - 20a_2 x^3 + 500x^2)(-6x)\, dx = 0.$$

$$\Rightarrow \int_0^{1/2} (36a_1 x^2 + 120a_2 x^4 - 3000x^3)\, dx = 0.$$

$$\Rightarrow \left[36a_1\left[\frac{x^3}{3}\right] + 120a_2\left[\frac{x^5}{5}\right] - 3000\left[\frac{x^4}{4}\right]\right]_0^{1/2} = 0.$$

$$a_1 + 0.5a_2 = 31.25 \quad \cdots\cdots \rightarrow \text{(E)}$$

* Domain 2

$$\frac{\partial I}{\partial a_2} = \int_{1/2}^1 R \cdot \frac{\partial R}{\partial a_2}\, dx.$$

$$R = -6a_1 x - 20a_2 x^3 + 500x^2$$

$$\frac{\partial R}{\partial a_2} = -20x^3.$$

$$\Rightarrow \frac{\partial \Pi}{\partial a_2} = \int_{1/2}^{1} (-6a_1 x - 20a_2 x^3 + 500x^2)(-20x^3)\, dx.$$

$$\frac{\partial \Pi}{\partial a_2} = 0,$$

$$\Rightarrow \int_{1/2}^{1} (120a_1 x^4 + 400 a_2 x^6 - 10000 x^5)\, dx = 0.$$

$$\Rightarrow \left[120a_1 \left(\frac{x^5}{5}\right) + 400 a_2 \left(\frac{x^7}{7}\right) - 10000 \left(\frac{x^6}{6}\right) \right]_{1/2}^{1} = 0$$

$$a_1 + 2.438 a_2 = 70.564 \dashrightarrow \text{(F)}$$

Solving (E) & (F)

$$a_1 = 21.11$$

$$a_2 = 20.28$$