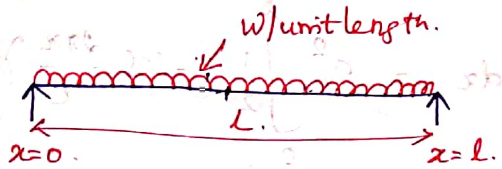
Rayleigh - Ritz Method.

Lecture - 4 – 22/09/2020.
- 5 – 24/09/2020.

Problems:

① A simply supported beam subjected to udl over entire span. Determine the bending moment and deflection at midspan by using Rayleigh - Ritz method & compare with exact solutions.



To Find:
(i) Deflection & Bending moment
(ii) Compare with exact soln.

Soln:-

For SSB, the fourier series,

$$Y = \sum_{n=1,3}^{\infty} a \sin \frac{n\pi x}{l} \quad \text{(approximating fn.)}$$

Let us consider

$$\text{Deflection}, \; y = a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{3\pi x}{l}. \quad [a_1, a_2 - \text{Ritz parameters}]$$

Total potential Energy of the beam, $\pi = U - H$

Strain Energy of the beam due to bending,

$$U = \frac{EI}{2} \int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx.$$

U – Strain Energy
H – Work done by ext. force

$$y = a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{3\pi x}{l}.$$

$$\frac{dy}{dx} = a_1 \cos \frac{\pi x}{l} \left(\frac{\pi}{l}\right) + a_2 \cos \frac{3\pi x}{l} \cdot \left(\frac{3\pi}{l}\right)$$

$$= \frac{a_1 \pi}{l} \cdot \cos \frac{\pi x}{l} + \frac{a_2 3\pi}{l} \cdot \cos \frac{3\pi x}{l}.$$

$$\frac{d^2y}{dx^2} = -\frac{a_1 \pi}{l} \sin \frac{\pi x}{l} \times \frac{\pi}{l} - a_2 \frac{3\pi}{l} \sin \frac{3\pi x}{l} \times \frac{3\pi}{l}.$$

$$\boxed{\frac{d^2y}{dx^2} = -\frac{\pi^2 a_1}{l^2} \sin \frac{\pi x}{l} - a_2 \frac{9\pi^2}{l^2} \sin \frac{3\pi x}{l}} \quad \cdots \to ①.$$

$$U = \frac{EI}{2} \int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx \cdots \to \text{sub. } ①$$

$$= \frac{EI}{2} \int_0^l \left[-\frac{a_1 \pi^2}{l^2} \sin \frac{\pi x}{l} - a_2 \frac{9\pi^2}{l^2} \sin \frac{3\pi x}{l}\right]^2 dx.$$

$$= \frac{EI}{2} \times \frac{\pi^4}{l^4} \int_0^l \left[a_1 \sin \frac{\pi x}{l} + 9 a_2 \sin \frac{3\pi x}{l}\right]^2 dx$$

$$\boxed{(a+b)^2 = a^2+b^2+2ab.}$$

$$U = \frac{EI}{2} \times \frac{\pi^4}{l^4} \int_0^l \left[a_1^2 \sin^2 \frac{\pi x}{l} + 81 a_2^2 \sin^2 \frac{3\pi x}{l} + 2\left(a_1 \sin \frac{\pi x}{l}\right)\left(9 a_2 \sin \frac{3\pi x}{l}\right)\right] dx$$

$$U = \frac{EI}{2} \frac{\pi^4}{l^4} \int_0^l \left[a_1^2 \sin^2 \frac{\pi x}{l} + 81 a_2^2 \sin^2 \frac{3\pi x}{l} + 18 a_1 a_2 \sin \frac{\pi x}{l} \sin \frac{3\pi x}{l}\right] dx$$

$$\int_0^l a_1^2 \sin^2 \frac{\pi x}{l} dx = a_1^2 \int_0^l \frac{1}{2}\left(1-\cos \frac{2\pi x}{l}\right) dx. \qquad \boxed{\sin^2 x = \frac{1-\cos 2x}{2}.}$$

$$= \frac{a_1^2}{2} \int_0^l \left(1 - \cos \frac{2\pi x}{l}\right) dx.$$

$$= \frac{a_1^2}{2} \left[\int_0^l dx - \int_0^l \cos \frac{2\pi x}{l} . dx\right]$$

$$= \frac{a_1^2}{2} \left[(x)_0^l - \left(\frac{\sin \frac{2\pi x}{l}}{\frac{2\pi}{l}}\right)_0^l\right]$$

$$= \frac{a_1^2}{2} \left[l - 0 - \frac{1}{2\pi}\left(\sin \frac{2\pi l}{l} - \sin 0\right)\right]$$

$$\boxed{\sin 2\pi = 0; \quad \sin 0 = 0}$$

$$\frac{a_1^2}{2}\left[l - \frac{1}{2\pi}(0-0)\right]$$

$$\boxed{\int_0^l a_1^2 \sin^2 \frac{\pi x}{l} dx = \frac{a_1^2 l}{2}}$$

$$= \frac{a_1^2 l}{2}.$$

$$\int_0^l 81 a_2^2 \sin^2 \frac{3\pi x}{l} = 81 a_2^2 \int_0^l \frac{1}{2}\left(1 - \cos \frac{6\pi x}{l}\right) \qquad \boxed{\sin^2 x = \frac{1-\cos 2x}{2}.}$$

$$= \frac{81 a_2^2}{2} \left[(x)_0^l - \left(\frac{\sin \frac{6\pi x}{l}}{\frac{6\pi}{l}}\right)_0^l\right]$$

$$= \frac{81 a_2^2}{2} \left[l - 0 - \frac{1}{6\pi}\left(\sin \frac{6\pi l}{l} - \sin 0\right)\right]$$

$$= \frac{81 a_2^2}{2} \left[l - \frac{1}{6\pi}(\sin 6\pi - \sin 0)\right] \qquad \boxed{\sin 6\pi = 0; \quad \sin 0 = 0}$$

$$\frac{81 a_2^2}{2}[l - 0]$$

$$= \frac{81 a_2^2 l}{2}.$$

$$\int_0^l 18 a_1 a_2 \sin\frac{\pi x}{l} \sin\frac{3\pi x}{l}.$$

$$= 18 a_1 a_2 \int_0^l \sin\frac{\pi x}{l} \sin\frac{3\pi x}{l}. = 18 a_1 a_2 \int_0^l \sin\frac{3\pi x}{l} \sin\frac{\pi x}{l}.$$

$$= \frac{18 a_1 a_2}{2}\left[\int_0^l \cos\frac{2\pi x}{l}dx - \int_0^l \cos\frac{4\pi x}{l}dx\right] \qquad \boxed{\sin A \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}}$$

$$= \frac{\overset{9}{\cancel{18}} a_1 a_2}{\cancel{2}}\left[\left(\frac{\sin\frac{2\pi x}{l}}{\frac{2\pi}{l}}\right)_0^l - \left(\frac{\sin\frac{4\pi x}{l}}{\frac{4\pi}{l}}\right)_0^l\right] \qquad \boxed{\begin{array}{l}\sin 2\pi = 0\\ \sin 4\pi = 0\\ \sin 0 = 0\end{array}}$$

$$= 9 a_1 a_2 [0 - 0].$$

$$= 0 //.$$

$$U = \frac{EI}{2}\frac{\pi^4}{l^4}\int_0^l \left[a_1^2 \sin^2\frac{\pi x}{l} + 81 a_2^2 \sin^2\frac{3\pi x}{l} + 18 a_1 a_2 \sin\frac{\pi x}{l}\sin\frac{3\pi x}{l}\right]dx \quad \dots \to (A)$$

Sub (1), (2), & (3) in (A)

$$U = \frac{EI}{2}\frac{\pi^4}{l^4}\left[\frac{a_1^2 l}{2} + \frac{81 a_2^2 l}{2} + 0\right]$$

$$= \frac{EI\pi^4 \cancel{l}}{4\, \cancel{l^4}^{\,3}}\left[a_1^2 + 81 a_2^2\right]$$

Strain Energy $U = \dfrac{EI\pi^4}{4 l^3}\left[a_1^2 + 81 a_2^2\right]$

work done by ext. force, $H = \displaystyle\int_0^l \omega y \, dx.$

$$= \int_0^l \omega\left(a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l}\right)dx$$

$$= \omega\int_0^l \left(a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l}\right)dx$$

$$= \omega\left[a_1\int_0^l \sin\frac{\pi x}{l}\,dx + a_2\int_0^l \sin\frac{3\pi x}{l}\,dx\right]$$

$$= \omega\left[a_1\left(\frac{-\cos\frac{\pi x}{l}}{\frac{\pi}{l}}\right)_0^l + a_2\left(\frac{-\cos\frac{3\pi x}{l}}{\frac{3\pi}{l}}\right)_0^l\right]$$

$$= \omega\left[-\frac{a_1 l}{\pi}\left(\cos\frac{\pi x}{l}\right)_0^l - \frac{a_2 l}{3\pi}\left(\cos\frac{3\pi x}{l}\right)_0^l\right]$$

$$= \omega\left[-\frac{a_1 l}{\pi}\left[(-1)-1\right] - \frac{a_2 l}{3\pi}(-1-1)\right]$$

$\cos 0 = 1$
$\cos\pi = -1$
$\cos 3\pi = -1$

$$= \omega\left[\frac{2a_1 l}{\pi} + \frac{2a_2 l}{3\pi}\right]$$

$$\boxed{H = \frac{2\omega l}{\pi}\left[a_1 + \frac{a_2}{3}\right]} \dashrightarrow (B)$$

$$\pi = U - H$$

$$\pi = \frac{EI\pi^4}{4l^3}(a_1^2 + 81a_2^2) - \frac{2\omega l}{\pi}\left[a_1 + \frac{a_2}{3}\right] \dashrightarrow (C)$$

for stationary value of π, the following condn. must be satisfied.

$$\frac{\partial\pi}{\partial a_1} = 0 \quad ; \quad \frac{\partial\pi}{\partial a_2} = 0$$

$$\Rightarrow \frac{\partial\pi}{\partial a_1} = \frac{EI\pi^4}{4l^3}(2a_1) - \frac{2\omega l}{\pi}(1)$$

$$\Rightarrow \frac{EI\pi^4}{4l^3}(2a_1) = \frac{2\omega l}{\pi}$$

$$= \frac{2\omega l}{\pi} \times \frac{4l^3}{EI\pi^4}$$

$$2a_1 = \frac{8\omega l^4}{EI\pi^5}$$

$$a_1 = \frac{4\omega l^4}{EI\pi^5}$$

$$\Rightarrow \boxed{a_1 = \frac{4\omega l^4}{EI\pi^5}} \dashrightarrow (4)$$

$$\Rightarrow \frac{\partial\pi}{\partial a_2} = \frac{EI\pi^4}{4l^3}(162a_2) - \frac{2\omega l}{\pi}\left(\frac{1}{3}\right) = 0.$$

$$\Rightarrow \frac{EI\pi^4}{4l^3}\cdot(162a_2^2) = \frac{2\omega l}{\pi}\left(\frac{1}{3}\right)$$

$$\Rightarrow a_2 = \frac{2\pi\omega l}{3\pi} \times \frac{4l^3}{162\, EI\pi^4}$$

$$\boxed{a_2 = \frac{4\omega l^3}{243\, EI\pi^5}} \dashrightarrow (5)$$

Sub a_1 & a_2 ((A) & (5)),

$$y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l}.$$

$$\Rightarrow \boxed{y = \frac{4\omega l^4}{EI\pi^5}\cdot \sin\frac{\pi x}{l} + \frac{4\omega l^3}{243\, EI\pi^5}\cdot \sin\frac{3\pi x}{l}}$$

max deflection occurs at $x = l/2$,

$$y_{max} = \frac{4\omega l^4}{EI\pi^5} \sin\frac{\pi(l/2)}{l} + \frac{4\omega l^4}{243\, EI\pi^5} \sin\frac{3\pi(l/2)}{l}$$

$$= \frac{4\omega l^4}{EI\pi^5} \sin\frac{\pi}{2} + \frac{4\omega l^4}{243\, EI\pi^5} \sin\frac{3\pi}{2}$$

$$= \frac{4\omega l^4}{EI\pi^5}(1) + \frac{4\omega l^4}{243\, EI\pi^5}(-1) \qquad \sin\frac{\pi}{2} = 1\ ;\ \sin\frac{3\pi}{2} = -1$$

$$y_{max} = \frac{4\omega l^4}{EI\pi^5} - \frac{4\omega l^4}{243\, EI\pi^5}$$

$$= \frac{4\omega l^4}{EI\pi^5}\left[1 - \frac{1}{243}\right]$$

$$= \frac{4\omega l^4}{EI\pi^5}(0.9958) \Rightarrow \frac{3.98\,\omega l^4}{EI\pi^5}$$

$$\boxed{y_{max} = 0.0130\,\frac{\omega l^4}{EI}}$$

Bending Moment at Midspan.

$$M = EI \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\left[\frac{a_1 \pi^2}{l^2} \sin \frac{\pi x}{l} + \frac{a_2 9\pi^2}{l^2} \sin \frac{3\pi x}{l}\right]$$

$$= -\left[\frac{4\omega l^4}{EI\pi^5} \cdot \frac{\pi^2}{l^2} \cdot \sin \frac{\pi x}{l} + \frac{4\omega l^4}{243 EI\pi^5} \cdot \frac{9\pi^2}{l^2} \cdot \sin \frac{3\pi x}{l}\right]$$

Max. Bending occurs at $x = l/2$

$$\Rightarrow \frac{d^2y}{dx^2} = -\left[\frac{4\omega l^4}{EI\pi^5} \times \frac{\pi^2}{l^2} \sin \frac{\pi\, l/2}{l} + \frac{4\omega l^4}{243 EI\pi^5} \cdot \frac{9\pi^2}{l^2} \cdot \sin \frac{3\pi\, l/2}{l}\right]$$

$\boxed{\sin \pi/2 = 1 \,;\ \sin 3\pi/2 = -1}$

$$= -\left[\frac{4\omega l^4}{EI\pi^5} \frac{\pi^2}{l^2} (1) + \frac{4\omega l^4}{243 EI\pi^5} \cdot \frac{9\pi^2}{l^2} (-1)\right]$$

$$= -\left[\frac{4\omega l^4 \pi^2}{EI\pi^5 l^2} - \frac{36\omega l^4 \pi^2}{243 EI\pi^5}\right]$$

$$= -\left[\frac{4\omega l^2 \pi^2}{EI\pi^5} - \frac{36\omega l^2 \pi^2}{243 EI\pi^5}\right]$$

$$= -\frac{4\omega l^2}{EI\pi^3} + \frac{36\omega l^2}{243 EI\pi^3}$$

$$= -\frac{4\omega l^2}{EI\pi^3} + \frac{0.148\omega l^2}{EI\pi^3} = -3.852 \frac{\omega l^2}{EI\pi^3}.$$

$$\frac{d^2y}{dx^2} = -0.124 \frac{\omega l^2}{EI}.$$

Bending Moment, $M = EI\left(\frac{d^2y}{dx^2}\right)$

$$M_{Centre} = EI \cdot (-0.124) \frac{\omega l^2}{EI}$$

$$\boxed{M_{Centre} = -0.124\, \omega l^2}$$ ∴ Obtained by R.R. method.

We know that

<u>Exact Solution:</u> $M_{Centre} = \frac{\omega l^2}{8} \Rightarrow 0.125\, \omega l^2$

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