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THE EXPONENTIAL DISTRIBUTION

A continuous random variable 'X' is said to follow an exponential distribution with parameter $\alpha > 0$ if its probability density function (p.d.f) is given by,

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Moment generating function:

$$M_X(t) = E(e^{tx})$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{0}^{\infty} e^{tx} . \alpha e^{-\alpha x}\, dx \quad [\because x \geq 0]$$

$$= \alpha \int_{0}^{\infty} e^{-(-t+\alpha) x}\, dx$$

$$= \alpha \left[\frac{e^{-(-t+\alpha)x}}{+t - \alpha} \right]_0^{\infty}$$

$$= \frac{\alpha}{+t - \alpha} \left[e^{-\infty} - e^{0} \right]$$

$$= \frac{\alpha}{+t - \alpha} [0 - 1]$$

$$= \frac{+\alpha}{-t+\alpha}$$

$$M_X(t) = \frac{\alpha}{\alpha - t}$$

Dividing numerator and denominator by 'α', we get,

$$M_X(t) = \frac{1}{1-\frac{t}{\alpha}}$$

$$= \left(1-\frac{t}{\alpha}\right)^{-1}$$

$$= 1 + \frac{t}{\alpha} + \frac{t^2}{\alpha^2} + \cdots + \frac{t^r}{\alpha^r} + \cdots$$

$$\boxed{M_X(t) = \sum_{r=0}^{\infty} \left(\frac{t}{\alpha}\right)^r}$$

$M_x(t) = \alpha(\alpha - t)^{-1}$

$M_x'(\alpha) = \alpha(-1)(\alpha-t)^{-2}(-$

$= \frac{\alpha}{(\alpha-t)^2}$

$M_x'(0) = \frac{\alpha}{\alpha^2} = \frac{1}{\alpha}$

$M_x''(t) = (-2)\alpha(\alpha-t)^{-3}(-1)$

$= 2\alpha(\alpha-t)^{-3}$

$M_x''(0) = \frac{2\alpha}{\alpha^3} = \frac{2}{\alpha^2}$

$Var = \left(\frac{2}{\alpha^2}\right) - \left(\frac{1}{\alpha}\right)^2 = \frac{2}{\alpha^2} - \frac{1}{\alpha^2}$

$= \frac{1}{\alpha^2}$

Mean and variance:

$$\mu_r' = E(X^r)$$

$$= \text{Coefficient of } \frac{t^r}{r!} \text{ in } M_X(t)$$

$$= \frac{r!}{\alpha^r}, \quad r = 1, 2, \cdots$$

$$\boxed{\text{Mean} = \mu_1' = \frac{1}{\alpha}}$$

$$\mu_2' = \frac{2!}{\alpha^2}$$

$$\text{Variance} = \mu_2' - (\mu_1')^2$$

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$$\text{Variance} = \frac{2}{\alpha^2} - \frac{1}{\alpha^2}$$

$$\boxed{\text{Variance} = \frac{1}{\alpha^2}}$$

Note :

$$\text{Standard Deviation} = \sqrt{\text{variance}} = \sqrt{\frac{1}{\alpha^2}} = \frac{1}{\alpha}$$

$\therefore$ For exponential distribution, $\boxed{\text{mean} = S.D = \frac{1}{\alpha}}$

Characteristic function :

$$\varphi_X(t) = E(e^{itx})$$

$$= \int_{-\infty}^{\infty} e^{itx} f(x)\, dx$$

$$= \int_{0}^{\infty} e^{itx} \alpha e^{-\alpha x}\, dx \qquad [\because x \geq 0]$$

$$= \alpha \int_{0}^{\infty} e^{(it-\alpha)x}\, dx$$

$$= \alpha \left[\frac{e^{-(-it+\alpha)x}}{-(-it+\alpha)} \right]_0^{\infty}$$

$$= \frac{\alpha}{it-\alpha} \left[e^{-\infty} - e^{0} \right]$$

$$= \frac{-\alpha}{it-\alpha}$$

$$= \frac{\alpha}{\alpha - it}$$

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If $0 < \alpha < 1$, Variance > Mean

$\alpha = 1$, Variance = Mean

$\alpha > 1$, Variance < Mean

Hence for an exponential distribution,

Variance >, =, or < Mean for different values of the parameter.

Exponential distribution lacks memory:

If X is exponentially distributed with parameter α, then for any two positive integers 's' and 't',

$$P[X > s+t \,/\, X > s] = P[X > t]$$

Proof:

The p.d.f of X is,

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Consider $P[X > t] = \int_t^\infty f(x)\,dx$

$$= \int_t^\infty \alpha e^{-\alpha x}\,dx$$

$$= \alpha \left[\frac{e^{-\alpha x}}{-\alpha}\right]_t^\infty$$

$$= -\left[e^{-\infty} - e^{-\alpha t}\right]$$

$$\therefore \quad P(X > t) = e^{-\alpha t} \longrightarrow (1)$$

Consider,

$$P[X > s+t / X > s] = \frac{P[X > s+t \cap X > s]}{P(X > s)}$$

$$= \frac{P(X > s+t)}{P(X > s)}$$

$$= \frac{e^{-\alpha(s+t)}}{e^{-s\alpha}} \quad (\text{using } (1))$$

$$= e^{-s\alpha}\, e^{-\alpha t} \cdot e^{s\alpha}$$

$$= e^{-\alpha t} \longrightarrow (2)$$

From (1) and (2),

$$\boxed{P[X > s+t / X > s] = P[X > t]}$$

Thus exponential distribution lacks memory.

Problems : [Memoryless Property of Exponential Distribution

(1) The mileage which car owners get with certain kind of radial tyre is a random variable having an exponential distribution with mean 4000 km. Find the probabilities that one of these tires will last

(i) atleast 2000 km (ii) atmost 3000 km.

Solution :

Given : $\alpha = \frac{1}{4000}$ Mean $= \frac{1}{\alpha} = \frac{1}{4000}$

Then $f(x) = \alpha e^{-\alpha x}$

$$f(x) = \frac{1}{4000} e^{-x/4000}, \quad x > 0$$

(i) Atleast 2000 km :

$$P[X > 2000] = \int_{2000}^{\infty} f(x)\,dx$$

$$= \int_{2000}^{\infty} \frac{1}{4000} e^{-x/4000}\,dx$$

$$= \frac{1}{4000}\left[\frac{e^{-x/4000}}{\frac{-1}{4000}}\right]_{2000}^{\infty}$$

$$= -\left[e^{\infty} - e^{-1/2}\right]$$

$$P[x > 2000] = e^{-0.5}$$

$$P[x > 2000] = 0.6065$$

(ii) Atmost 3000 km :

$$P(X \leq 3000) = \int_0^{3000} f(x)\,dx$$

$$= \int_0^{3000} \frac{1}{4000} e^{-x/4000}\,dx$$

$$= \frac{1}{4000}\left[\frac{e^{-x/4000}}{-1/4000}\right]_0^{3000}$$

$$= -\left[-e^0 + e^{-3/4}\right]$$

$$= 1 - e^{-0.75}$$

$$P(X \leq 3000) = 0.5270.$$

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(2) For an exponential distribution with mean 120 days, find the probability that such a watch will
(i) have to be set in less than 24 days and
(ii) not have to be reset in atleast 180 days.

Solution :

Given : Mean $= \frac{1}{\alpha} = 120$

$$\Rightarrow \alpha = \frac{1}{120}$$