POISSON DISTRIBUTION

Definition: A random variable X is said to follow Poisson distribution if it assumes only non-negative values and its probability mass function is given by,

$$P(X=x) = p(x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{x}}{x!}, & x = 0, 1, 2, \dots \infty \\ 0, & \text{otherwise.} \end{cases}$$

Moment Generating function:

$$M_X(t) = E(e^{tx})$$

$$= \sum_{x=0}^{\infty} e^{tx}\, p(x)$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{x!} e^{tx}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^{t})^{x}}{x!}$$

$$= e^{-\lambda}\left[1 + \frac{\lambda e^{t}}{1!} + \frac{(\lambda e^{t})^{2}}{2!} + \cdots\right]$$

$$= e^{-\lambda} e^{\lambda e^{t}} \qquad \left(\because e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)$$

$$\boxed{M_X(t) = e^{\lambda(e^{t}-1)}}$$

Mean and Variance:

$$\mu_1' = E(X)$$

$$= \sum_{x=0}^{\infty} x \cdot p(x)$$

$$= \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= e^{-\lambda}\left[0 + 1 \cdot \frac{\lambda}{1!} + 2 \cdot \frac{\lambda^{2}}{2!} + 3\frac{\lambda^{3}}{3!} + \cdots\right]$$

$$\mu_1 = \lambda e^{-\lambda}\left[1+\frac{\lambda}{1!}+\frac{\lambda}{2!}+\frac{\lambda^2}{3!}+\cdots\right]$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$\boxed{\mu_1' = \text{Mean} = \lambda}$$

$$\mu_2' = E(X^2)$$

$$= \sum_{x=0}^{\infty} x^2 \cdot p(x)$$

$$= \sum_{x=0}^{\infty} x^2 \cdot \frac{e^{-\lambda}\cdot\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} [x(x-1)+x]\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty} x\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{\cancel{x}(\cancel{x-1})e^{-\lambda}\lambda^{x-2}\lambda^2}{\cancel{x}(\cancel{x-1})(x-2)\cdots 1} + \lambda \quad (\because \mu_1' = \lambda)$$

$$= e^{-\lambda}\lambda^2 \sum_{x=0}^{\infty}\frac{\lambda^{x-2}}{(x-2)(x-3)\cdots 1} + \lambda$$

$$= e^{-\lambda}\lambda^2 \sum_{x=2}^{\infty}\frac{\lambda^{x-2}}{(x-2)!} + \lambda$$

$$= e^{-\lambda}\lambda^2\left[1+\frac{\lambda}{1!}+\frac{\lambda^2}{2!}+\cdots\right]+\lambda$$

$$= +e^{-\lambda}\lambda^2 e^{\lambda} + \lambda$$

$$\boxed{\mu_2' = \lambda^2+\lambda}$$

$$\text{Variance} = \mu_2' - \mu_1'^2$$

$$= \lambda^2+\lambda-\lambda^2 = \lambda$$

$$\boxed{\text{Variance} = \lambda}$$

(or)

Mean:

$$M_X(t) = e^{\lambda(e^t - 1)} = e^{\lambda e^t} \cdot e^{-\lambda}$$

$$M_X'(t) = \lambda e^t e^{\lambda e^t} e^{-\lambda}$$

$$M_X'(0) = \lambda \cdot e^{\lambda} \cdot e^{-\lambda}$$

$$M_X'(0) = \lambda$$

$$\boxed{\text{Mean} = E(X) = M_X'(0) = \lambda}$$

$$M_X''(t) = E(X^2) = (\lambda e^t)^2 e^{\lambda e^t} e^{-\lambda} + \lambda e^{\lambda e^t} e^{-}$$

$$M_X''(0) = \lambda^2 e^{\lambda} e^{-\lambda} + \lambda e^{\lambda} e^{-\lambda}$$

$$E(X^2) = M_X''(0) = \lambda^2 + \lambda$$

$$\text{Variance} = E(X^2) - [E(X)]^2$$

$$= \lambda^2 - \lambda^2 + \lambda$$

$$\boxed{\text{Variance} = \lambda}$$

Prove that poisson distribution is the limiting case of binomial distribution.

Suppose in a binomial distribution,

1. The number of trials is indefinitely large i.e., $n \to \infty$
2. p is very small i.e., $p \to 0$
3. $np = \lambda$ is finite.

Now $P(X = x) = nC_x \, p^x q^{n-x}$, $x = 0, 1, 2, \ldots n$

$$= \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!} p^x q^{n-x}$$

$$= \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!}\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}\right]$$

Taking limit as $n \to \infty$,

$$\lim_{n\to\infty} p(x) = \frac{\lambda^x}{x!} e^{-\lambda} \text{ for } x = 0, 1, 2, \ldots$$

$$\left(\because \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}\right)$$

which is the p.m.f of the poisson distribution.

Problems :

① If X is a poisson Variate,
$P(X=2) = 9P(X=4) + 90\,P(X=6)$,
find (i) mean of X (ii) Variance of X ~~(iii) $P(X \geq 2)$~~ ~~(iv) $E(X^2)$~~.

Solution:

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x = 0,1,2,\dots$$

Given: $P(X=2) = 9P(X=4) + 90\,P(X=6)$

i.e., $$\frac{e^{-\lambda}\lambda^{2}}{2!} = 9\,\frac{e^{-\lambda}\lambda^{4}}{4!} + 90\,\frac{e^{-\lambda}\lambda^{6}}{6!}$$

$$\frac{1}{2}\left(\cancel{e^{-\lambda}}\cancel{\lambda^{2}}\right) = \cancel{e^{-\lambda}}\cancel{\lambda^{2}}\left(\frac{9\lambda^{2}}{4!} + \frac{90\lambda^{4}}{6!}\right)$$

$$\frac{1}{2} = \frac{9\lambda^{2}}{4!} + \frac{90\lambda^{4}}{6!}$$

$$\frac{1}{2} = \frac{3\lambda^{2}}{8} + \frac{\lambda^{4}}{8}$$

$$1 = \frac{3\lambda^{2}}{4} + \frac{\lambda^{4}}{4}$$

$$4 = \lambda^{4} + 3\lambda^{2}$$

$$\lambda^{4} + 3\lambda^{2} - 4 = 0$$

$$\lambda^{2} = \frac{-3 \pm \sqrt{9+16}}{2} = \frac{-3 \pm 5}{2} = 1 \text{ or } -4$$

$\lambda^{2} = 1 \Rightarrow \lambda = \pm 1$ $\quad$ $\lambda = -1$ is also not possible.

$\lambda^{2} = -4$ is not possible.

$\therefore$ Mean $= \lambda = 1$

Variance $= \lambda = 1$

(2) The number of monthly breakdown of a computer is a random variable having a poisson distribution with mean equal to 1.8. Find the probability that this computer will function for a month (i) with only one breakdown (ii) without a breakdown (iii) with atleast one breakdown.

Solution:

Let X denotes the number of breakdowns in a month.

$$P(X=x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$$

Given: Mean $= \lambda = 1.8$.

$$\therefore P(X=x) = \frac{e^{-1.8} (1.8)^{x}}{x!}$$

(i) P(with only one breakdown)

$$= P(X=1)$$

$$= \frac{e^{-1.8} (1.8)^{1}}{1!}$$

$$= 0.2975$$

(ii) P (without a break down) $= P(X = 0)$

$$= \frac{e^{-1.8}(1.8)^0}{0!} = e^{-1.8}$$

$$= 0.1653$$

(iii) P (Atleast one breakdown) $= P(X \geq 1)$

$$= 1 - P(X < 1)$$

$$= 1 - P(X = 0)$$

$$= 1 - 0.1653$$

$$= 0.8347.$$

(3) In a certain factory turning razar blades there is a small chance of 1/500 for any blade to be defective. The blades are in packets of 10. Use Poisson distribution to calculate the approximate number of packets containing (i) no defective (ii) One defective (iii) 2 defective blades respectively in a consignment of 10,000 packets.

Solution :

Let X denote the number of defective blades.

Given : $p = \frac{1}{500}$, $n = 10$

$N = 10,000$

$$\text{Mean} = \lambda = np = 10 \times \frac{1}{500} = 0.02$$

$$\lambda = 0.02$$

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \frac{e^{-0.02}(0.02)^x}{x!}$$

(i) P(no defective blades) $= P(X=0)$

$$= \frac{e^{-0.02}(0.02)^0}{0!} = e^{-0.02}$$

$$= 0.9802$$

$\therefore$ Total number of packets containing no defective blades in 10,000 packets

$= N \times P(\text{no defective})$

$= 10,000 \times 0.9802$

$= \underline{9802 \text{ packets}}$.

(ii) P(one defective blade) $= P(X=1)$

$$= \frac{e^{-0.02}(0.02)^1}{1!}$$

$$= 0.01960$$

Number of packets containing one defective blades

$= N \times P(\text{one defective})$

$= 10,000 \times 0.01960$

$= \underline{196 \text{ packets}}$.

(iii) P(two defective blades) $= P(X=2)$

$$= \frac{e^{-0.02}(0.02)^2}{2!}$$

$$= 0.000196$$

Number of packets containing two defective blades

$= N \times P(\text{two defectives})$

$= 10,000 \times 0.000196$

$\simeq \underline{2 \text{ packets}}$.

HW (4) The average number of traffic accidents on a certain section of a highway is two per week. Assume that the number of accidents follow a poisson distribution. Find the probability of (i) no accident in a week (ii) atmost two accidents in a 2 week period

Solution:

No of accidents per week, $\lambda = 2$

$$P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= \frac{e^{-2} 2^x}{x!}$$

(i) P(no accidents) $= P(X=0) = \frac{e^{-2} 2^0}{0!} = e^{-2}$

$= 0.1353.$

(ii) During a 2 week period the average number of accidents $= 2+2 = 4$. Here $\lambda = 4$

$\therefore$ P(atmost ~~2~~ 4 accidents during 2-week period) $= P(X<4)$

$$= P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= \frac{e^{-4} 4^0}{0!} + \frac{e^{-2} 4^1}{1!} + \frac{e^{-2} 4^2}{2!} + \frac{e^{-2} 4^3}{3!}$$

$$= \frac{e^{-4} 4^0}{0!} + \frac{e^{-4} 4^1}{1!} + \frac{e^{-4} 4^2}{2!} + \frac{e^{-4} 4^3}{3!}$$

$$= e^{-4}\left[1 + 4 + \frac{16}{2} + \frac{64}{6}\right]$$

$$= e^{-4}(23.67)$$

$$= 0.4335.$$

(5) If X and Y are independent random variables. Show that the conditional distribution of X given X+Y is a binomial distribution.

Solution:

Let X and Y be independent random variables with parameters λ_1 & λ_2.

$$P(X = r / X+Y = n) = \frac{P(X = r \cap X+Y = n)}{P(X+Y = n)}$$

$$= \frac{P(X = r \cap Y = n-r)}{P(X+Y = n)}$$

$$= \frac{P(X = r) \cdot P(Y = n-r)}{P(X+Y = n)}$$

$$= \frac{\frac{e^{-\lambda_1} \lambda_1^r}{r!} \cdot \frac{e^{-\lambda_2} \cdot \lambda_2^{n-r}}{(n-r)!}}{\frac{e^{-(\lambda_1+\lambda_2)} (\lambda_1+\lambda_2)^n}{n!}}$$

$$= \frac{n!}{r!(n-r)!} \frac{\lambda_1^r \lambda_2^{n-r}}{(\lambda_1+\lambda_2)^{n-r+r}}$$

$$= nC_r \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^r \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-r}$$

$$= nC_r \, p^r q^{n-r} \quad \text{where } p = \frac{\lambda_1}{\lambda_1+\lambda_2}$$

$$q = \frac{\lambda_2}{\lambda_1+\lambda_2}$$