3) Design a linear phase FIR bandpass filter to pass frequencies in the range 0.4π to 0.65π rad/sample by taking 7 samples of hanning window sequence.

Soln :- choose symmetric impulse response with symmetry condition $h(N-1-n) = h(n)$

The desired ideal frequency response $H_d(e^{j\omega})$ for bandpass filter is

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha}, & -\omega_{c2} \le \omega \le -\omega_{c1} \text{ \& } \omega_{c1} \le \omega \le \omega_{c2} \\ 0, & \text{otherwise} \end{cases}$$

The desired impulse response $h_d(n)$ is obtained by taking inverse fourier transform of $H_d(e^{j\omega})$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}} e^{-j\omega\alpha} e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_{c1}}^{\omega_{c2}} e^{-j\omega\alpha} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}} e^{j\omega(n-\alpha)}\, d\omega + \frac{1}{2\pi}\int_{\omega_{c1}}^{\omega_{c2}} e^{j\omega(n-\alpha)}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\omega_{c2}}^{-\omega_{c1}} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{\omega_{c1}}^{\omega_{c2}}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega_{c1}(n-\alpha)}}{j(n-\alpha)} - \frac{e^{-j\omega_{c2}(n-\alpha)}}{j(n-\alpha)}\right] + \frac{1}{2\pi}\left[\frac{e^{j\omega_{c2}(n-\alpha)}}{j(n-\alpha)} - \frac{e^{j\omega_{c1}(n-\alpha)}}{j(n-\alpha)}\right]$$

$$= \frac{1}{\pi(n-\alpha)}\left[\frac{e^{j\omega_{c2}(n-\alpha)} - e^{-j\omega_{c2}(n-\alpha)}}{2j} - \frac{e^{j\omega_{c1}(n-\alpha)} - e^{-j\omega_{c1}(n-\alpha)}}{2j}\right]$$

$$h_d(n) = \frac{\sin w_{c2}(n-\alpha) - \sin w_{c1}(n-\alpha)}{\pi(n-\alpha)} \text{ ; for all } n \text{ except } n=\alpha$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

when $n=\alpha$; $h_d(n) = \underset{n-\alpha \to 0}{Lt} \dfrac{\sin w_{c2}(n-\alpha) - \sin w_{c1}(n-\alpha)}{\pi(n-\alpha)}$

$$= \frac{1}{\pi}\left[\underset{(n-\alpha)\to 0}{Lt} \frac{\sin w_{c2}(n-\alpha)}{n-\alpha} - \underset{(n-\alpha)\to 0}{Lt} \frac{\sin w_{c1}(n-\alpha)}{n-\alpha} \right]$$

$$h_d(n) = \frac{w_{c2} - w_{c1}}{\pi}$$

Using L' Hopital rule $\underset{\theta \to 0}{Lt} \dfrac{\sin A\theta}{\theta} = A$

The impulse response $h(n)$ of FIR filter is obtained by multiplying $h_d(n)$ by window sequence

The hanning window sequence $w_c(n)$ is given by

$$W_c(n) = \begin{cases} 0.5 - 0.5\cos\frac{2\pi n}{N-1} & \text{for } n=0 \text{ to } N-1 \\ 0 & \text{; otherwise} \end{cases}$$

$$\therefore h(n) = h_d(n)\, W_c(n) = \left[\frac{\sin w_{c2}(n-\alpha) - \sin w_{c1}(n-\alpha)}{\pi(n-\alpha)}\right]\left[0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right)\right] \text{ for } n \neq \alpha$$

$$= \left(\frac{w_{c2} - w_{c1}}{\pi}\right)\left[0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right)\right] \text{ for } n = \alpha$$

Given that $N=7$ $w_{c1} = 0.4\pi$ rad/sample and $w_{c2} = 0.65\pi$ rad/sample

$\alpha = \dfrac{N-1}{2} = \dfrac{7-1}{2} = 3$; $N-1 = 6$ $\therefore h(n)$ for $n=0$ to 6

$h(n)$ satisfies the symmetry condition $h(N-1-n) = h(n)$

$$\therefore h(n) = \left[\frac{\sin w_{c2}(n-3) - \sin w_{c1}(n-3)}{\pi(n-3)}\right]\left[0.5 - 0.5\cos\frac{n\pi}{3}\right] \text{ for } n \neq 3$$

$$h(n) = \left(\frac{w_{c2} - w_{c1}}{\pi}\right)\left(0.5 - 0.5\cos\frac{n\pi}{3}\right) \text{ for } n=3$$

when $n=0$

$$h(0) = \frac{\left[\sin\left(0.65\pi(0-3)\right) - \sin\left(0.4\pi(0-3)\right)\right]\left[0.5 - 0.5\cos\frac{0\times\pi}{3}\right]}{\pi\,(0-3)} = 0$$

when $n=1$

$$h(1) = \frac{\left[\sin\left(0.65\pi(1-3)\right) - \sin\left(0.4\pi(1-3)\right)\right]\left[0.5 - 0.5\cos\frac{1\times\pi}{3}\right]}{\pi\,(1-3)} = -0.0556$$

when $n=2$

$$h(2) = \frac{\left[\sin\left(0.65\pi(2-3)\right) - \sin\left(0.4\pi(2-3)\right)\right]\left[0.5 - 0.5\cos\frac{2\times\pi}{3}\right]}{\pi\,(2-3)} = -0.0143$$

when $n=3$

$$h(3) = \left(\frac{0.65\pi - 0.4\pi}{\pi}\right)\left(0.5 - 0.5\cos\frac{3\pi}{3}\right) = 0.25$$

when $n=4$; $h(4) = h(6-4) = h(2) = -0.0143$

when $n=5$; $h(5) = h(6-5) = h(1) = -0.0556$

when $n=6$; $h(6) = h(6-6) = h(0) = 0$

The transfer function $H(z)$ of FIR bandpass filter is given by

$$H(z) = z\{h(n)\} = \sum_{n=0}^{N-1} h(n)\,z^{-n} \Rightarrow \sum_{n=0}^{6} h(n)\,z^{-n}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(2)z^{-4} + h(1)z^{-5} + h(0)z^{-6}$$

$$= h(0)\left[1+z^{-6}\right] + h(1)\left[z^{-1}+z^{-5}\right] + h(2)\left[z^{-2}+z^{-4}\right] + h(3)z^{-3}$$

$$= 0\left[1+z^{-6}\right] - 0.0556\left[z^{-1}+z^{-5}\right] - 0.0143\left[z^{-2}+z^{-4}\right] + 0.25z^{-3}$$

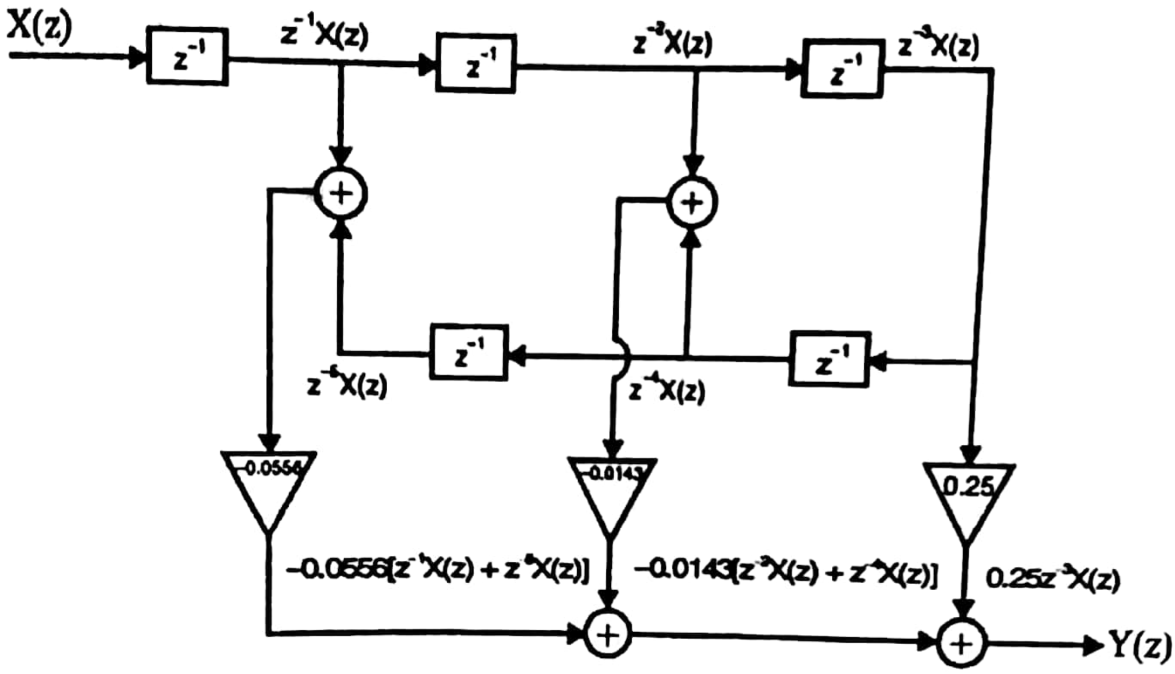
$$H(z) = -0.0556\left[z^{-1}+z^{-5}\right] - 0.0143\left[z^{-2}+z^{-4}\right] + 0.25z^{-3}$$

Structure :- $H(z) = \dfrac{Y(z)}{X(z)}$

$$H(z) = \frac{Y(z)}{X(z)} = -0.0556\left[z^{-1}+z^{-5}\right] - 0.0143\left[z^{-2}+z^{-4}\right] + 0.25z^{-3}$$

$$Y(z) = -0.0556\left[z^{-1}X(z)+z^{-5}X(z)\right] - 0.0143\left[z^{-2}X(z) + z^{-4}X(z)\right] + 0.25\,z^{-3}X(z)$$

Linear phase structure for FIR Bandpass filter :-



Frequency Response :-

$$A(\omega) = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right)\cos\omega n$$

$$A(\omega) = h(3) + \sum_{n=1}^{3} 2h(3-n)\cos\omega n$$

$$= h(3) + 2h(2)\cos\omega(1) + 2h(1)\cos 2\omega + 2h(0)\cos 3\omega$$

$$= 0.25 + 2\times(-0.0143)\cos\omega + (2\times -0.0556)\cos 2\omega + 2\times 0\cos 3\omega$$

$$A(\omega) = 0.25 - 0.0286\cos\omega - 0.1112\cos 2\omega$$