SNS COLLEGE OF TECHNOLOGY

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Coimbatore-35

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DEPARTMENT OF ELECTRONICS & COMMUNICATION ENGINEERING

19ECB212 – DIGITAL SIGNAL PROCESSING

II YEAR/ IV SEMESTER

UNIT 2 – IIR FILTER DESIGN

TOPIC – BUTTERWORTH FILTER

DESIGN OF LOWPASS DIGITAL BUTTERWORTH FILTER

- The popular methods of designing IIR digital filter involves the design of equivalent analog filter and then converting the analog filter into digital filter
- Hence to design a Butterworth IIR digital filter, first an analog butterworth filter transfer function is determined using the given specifications
- Then the analog filter transfer function is converted to a digital filter transfer function by using either Impulse Invariant Transformation (or) Bilinear Transformation

ANALOG BUTTERWORTH FILTER

- The analog Butterworth filter is designed by approximating the ideal analog filter frequency response, $H(j\Omega)$ using an error function
- The error function is selected such that the magnitude is maximally flat in the passband and monotonically decreasing in the stopband (The magnitude is maximally flat at the origin i.e., $\Omega = 0$ and monotonically decreasing with increasing Ω
- The magnitude response of lowpass filter obtained by this approximation is given by $$|H(\Omega)|^2 = \frac{1}{1+\left[\frac{\Omega}{\Omega c}\right]^{2N}}$$

PROPERTIES OF BUTTERWORTH FILTERS

- The Butterworth filters are all pole designs (i.e., the zeros of the filters exist at infinity)
- At the cutoff frequency Ω_c the magnitude of normalized Butterworth filter is $1/\sqrt{2}$ (i.e., $|H(j\Omega)| = 1/\sqrt{2} = 0.707$) Hence the dB magnitude at the cutoff frequency will be 3 dB less than the maximum value
- The filter order N completely specifies the filter
- The magnitude is maximally flat at the origin
- The magnitude is a monotonically decreasing function of Ω
- The magnitude response approaches the ideal response as the value of N increases

TRANSFER FUNCTION OF ANALOG BUTTERWORTH LOWPASS FILTER

- For a stable and causal filter the poles should lie on the left half of s-plane. Hence the digital filter transfer function is formed by choosing the N – number of left half poles
- When N is even, all the poles are complex and exist as conjugate pair. When N is odd, one of the poles is real and all other poles are complex and exist as conjugate pair
- Therefore the transfer function of Butterworth filters will be a product of second order factors

NORMALIZED BUTTERWORTH LPF TRANSFER FUNCTION

- N be the order of the filter
- $H(s_n)$ be the normalized Butterworth lowpass filter function
- When N is even,

$$H(s_n) = \prod_{k=1}^{\frac{N}{2}} \frac{1}{s_n^2 + b_k s_n + 1}$$

- When N is odd,

$$H(s_n) = \frac{1}{s_n + 1} \prod_{k=1}^{\frac{N-1}{2}} \frac{1}{s_n^2 + b_k s_n + 1}$$

$$\text{where, } b_k = 2 \sin\left[\frac{(2k-1)\pi}{2N}\right]$$

UNNORMALIZED BUTTERWORTH LPF TRANSFER FUNCTION

- The unnormalized transfer function is obtained by replacing s_n by $s/\ \Omega_c$ in the normalized transfer function, where Ω_c is the 3 dB cutoff frequency of the lowpass filter
- H(s) be the normalized Butterworth lowpass filter function
- When N is even,

$$\therefore\ H(s) = \prod_{k=1}^{\frac{N}{2}} \frac{1}{s_n^2 + b_k s_n + 1}\Bigg|_{s_n = \frac{s}{\Omega_c}}$$

$$= \prod_{k=1}^{\frac{N}{2}} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

UNNORMALIZED BUTTERWORTH LPF TRANSFER FUNCTION

- H(s) be the normalized Butterworth lowpass filter function
- When N is odd, H(s) is obtained by letting $s_n \rightarrow s/\Omega_c$ in the normalized Butterworth lowpass filter function

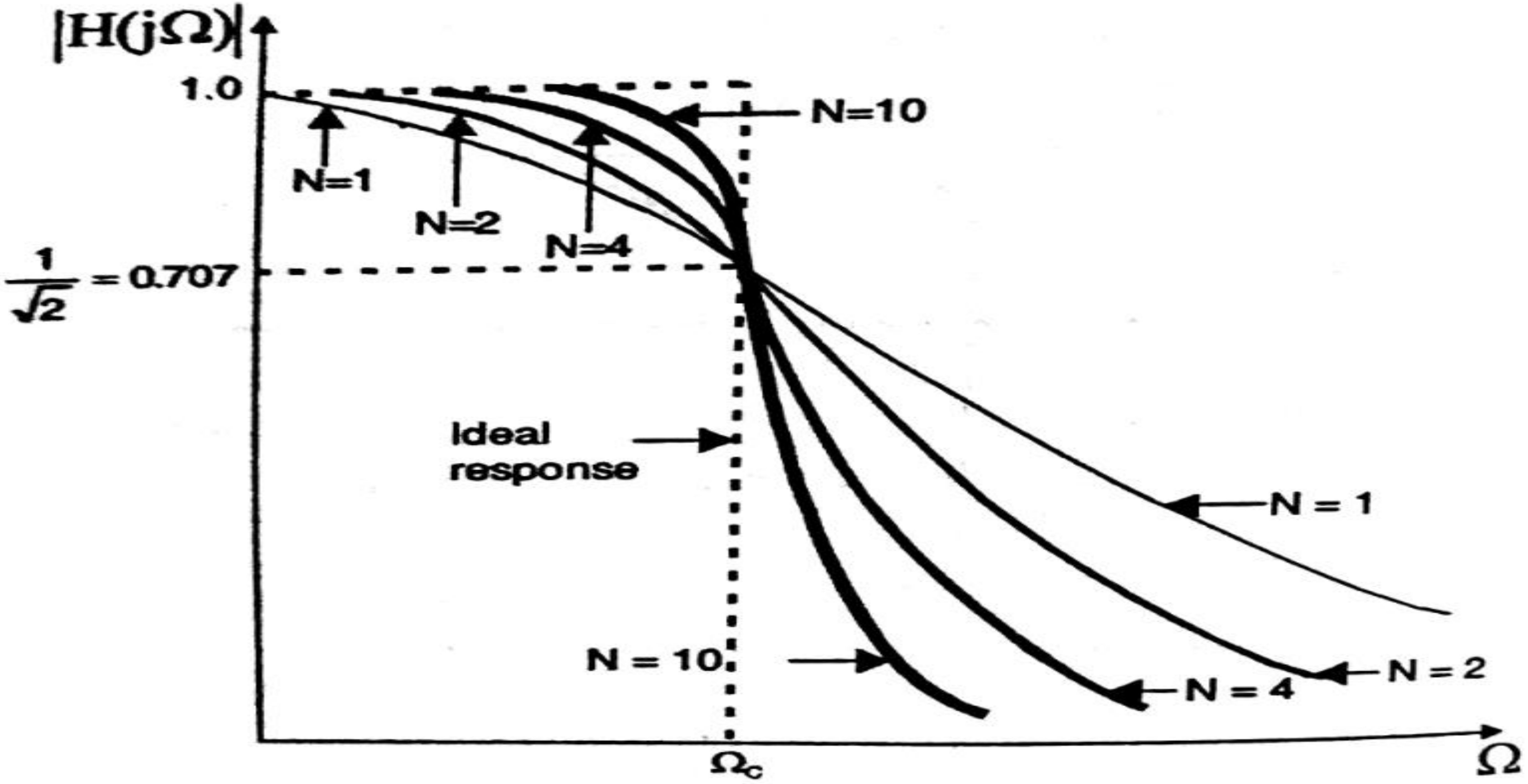
$$\therefore H(s) = \frac{1}{s_n + 1} \prod_{k=1}^{\frac{N-1}{2}} \frac{1}{s_n^2 + b_k s_n + 1} \Bigg|_{s_n = \frac{s}{\Omega_c}}$$

$$= \frac{\Omega_c}{s + \Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{s^2 + b_k \Omega_c + \Omega_c^2}$$

BUTTERWORTH LPF NORMALIZED TRANSFER FUNCTION

Order, N	Normalized tansfer function, $H(s_n)$
1	$\frac{1}{s_n+1}$
2	$\frac{1}{s_n^2+1.414s_n+1}$
3	$\frac{1}{(s_n+1)(s_n^2+s_n+1)}$
4	$\frac{1}{(s_n^2+0.765s_n+1)(s_n^2+1.848s_n+1)}$
5	$\frac{1}{(s_n+1)(s_n^2+0.618s_n+1)(s_n^2+1.618s_n+1)}$
6	$\frac{1}{(s_n^2+1.932s_n+1)(s_n^2+1.414s_n+1)(s_n^2+0.518s_n+1)}$

FREQUENCY RESPONSE OF ANALOG LOWPASS BUTTERWORTH FILTER



ORDER OF THE LOWPASS BUTTERWORTH FILTER

- In Butterworth filters the frequency response of the filter depends on the order N. The specifications of the filter are given in terms of gain at a passband and stopband frequency
- $\mathbf{A_p}$ - Gain or Magnitude at pass band edge frequency $\mathbf{\Omega_p}$
- $\mathbf{A_s}$ - Gain or Magnitude at Stop band edge frequency $\mathbf{\Omega_s}$

$$N_1 = \frac{1}{2}\frac{\log\left[\dfrac{(1/A_s^2) - 1}{(1/A_p^2) - 1}\right]}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

ORDER OF THE LOWPASS BUTTERWORTH FILTER

- The specifications of the filter are given in terms of dB attenuation at a passband and stopband frequency
- $\alpha_{p,\,dB}$ - dB attenuation at pass band edge frequency Ω_p
- $\alpha_{s,\,dB}$ - dB attenuation at Stop band edge frequency Ω_s

$$N_1 = \frac{\log\left[\left(\frac{10^{0.1\alpha_{s,dB}}-1}{10^{0.1\alpha_{p,dB}}-1}\right)^{\frac{1}{2}}\right]}{\log\frac{\Omega_s}{\Omega_p}}$$

LOWPASS BUTTERWORTH FILTER

- **Bilinear Transformation:**

$$\Omega_p = \frac{2}{T} \tan\frac{\omega_p}{2} \qquad \Omega_s = \frac{2}{T} \tan\frac{\omega_s}{2}$$

- **Impulse Invariant Transformation:**

$$\Omega_p = \frac{\omega_p}{T} \qquad \Omega_s = \frac{\omega_s}{T}$$

CUTOFF FREQUENCY OF LOWPASS BUTTERWORTH FILTER

- When the specifications are $\mathbf{A_p}$, $\mathbf{A_s}$, $\boldsymbol{\omega_p}$, $\boldsymbol{\omega_s}$

$$\text{Cutoff frequency, } \Omega_c = \frac{\Omega_s}{\left[\left(1/A_s^2\right)-1\right]^{\frac{1}{2N}}}$$

$$\text{Cutoff frequency, } \Omega_c = \frac{\Omega_p}{\left[\left(1/A_p^2\right)-1\right]^{\frac{1}{2N}}}$$

CUTOFF FREQUENCY OF LOWPASS BUTTERWORTH FILTER

- When the specifications are $\alpha_{p,\,dB}$, $\alpha_{s,\,dB}$, ω_p , ω_s

$$\text{Cutoff frequency, } \Omega_c = \frac{\Omega_s}{\left(10^{0.1\alpha_{s,dB}} - 1\right)^{\frac{1}{2N}}}$$

$$\text{Cutoff frequency, } \Omega_c = \frac{\Omega_p}{\left(10^{0.1\alpha_{p,dB}} - 1\right)^{\frac{1}{2N}}}$$

DESIGN PROCEDURE FOR LOWPASS DIGITAL BUTTERWORTH IIR FILTER

- $\boldsymbol{\omega_p}$ - Pass band edge digital frequency in rad /sample
- $\boldsymbol{\omega_s}$ - Stop band edge digital frequency in rad/sample
- $\mathbf{A_p}$ - Gain at pass band edge frequency $\boldsymbol{\omega_p}$
- $\mathbf{A_s}$ - Gain at Stop band edge frequency $\boldsymbol{\omega_s}$
- $\mathbf{T = 1/ F_s}$ - Sampling time in sec.
- Where F_s = Sampling frequency in Hz
- $\boldsymbol{\Omega_p}$ - Pass band edge analog frequency corresponding to $\boldsymbol{\omega_p}$
- $\boldsymbol{\Omega_s}$ - Stop band edge analog frequency corresponding to $\boldsymbol{\omega_s}$

DESIGN PROCEDURE FOR LOWPASS DIGITAL BUTTERWORTH IIR FILTER

1. Choose either Bilinear or Impulse Invariant transformation and determine the specifications of equivalent analog filter

- The gain or attenuation of analog filter is same as digital filter
- **Bilinear Transformation:**

$$\Omega_p = \frac{2}{T} \tan\frac{\omega_p}{2} \qquad \Omega_s = \frac{2}{T} \tan\frac{\omega_s}{2}$$

- **Impulse Invariant Transformation:**

$$\Omega_p = \frac{\omega_p}{T} \qquad \Omega_s = \frac{\omega_s}{T}$$

ORDER OF THE LOWPASS DIGITAL BUTTERWORTH FILTER

2. Decide the order N of the filter. In order to estimate the order N, Calculate the Parameter N_1 using the following equation:

$$N_1 = \frac{1}{2} \frac{\log\left[\frac{(1/A_s^2) - 1}{(1/A_p^2) - 1}\right]}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

- Choose N such that, $N \geq N_1$, Usually N is chosen as nearest integer just greater than N_1

3. Determine the normalized transfer function $H(s_n)$ of the analog lowpass filter function

- When N is even,

$$H(s_n) = \prod_{k=1}^{\frac{N}{2}} \frac{1}{s_n^2 + b_k s_n + 1}$$

- When N is odd,

$$H(s_n) = \frac{1}{s_n + 1} \prod_{k=1}^{\frac{N-1}{2}} \frac{1}{s_n^2 + b_k s_n + 1}$$

$$\text{where, } b_k = 2 \sin\left[\frac{(2k-1)\pi}{2N}\right]$$

CUTOFF FREQUENCY

4. Calculate the analog Cutoff frequency Ω_c

$$\text{Cutoff frequency, } \Omega_c = \frac{\Omega_s}{\left[\left(1/A_s^2\right)-1\right]^{\frac{1}{2N}}}$$

UNNORMALIZED ANALOG TRANSFER FUNCTION

5. Determine the unnormalized analog transfer function H (s) is obtained by replacing s_n by s/Ω_c in the normalized transfer function of the low pass filter function

- When N is even,

$$\therefore H(s) = \prod_{k=1}^{\frac{N}{2}} \frac{1}{s_n^2 + b_k s_n + 1}\Bigg|_{s_n = \frac{s}{\Omega_c}}$$

$$= \prod_{k=1}^{\frac{N}{2}} \frac{\Omega_c^2}{s^2 + b_k \Omega_c s + \Omega_c^2}$$

UNNORMALIZED ANALOG TRANSFER FUNCTION

- H(s) be the normalized Butterworth lowpass filter function
- When N is odd, H(s) is obtained by letting $s_n \rightarrow s/\Omega_c$ in the normalized Butterworth lowpass filter function

$$\therefore H(s) = \frac{1}{s_n + 1} \prod_{k=1}^{\frac{N-1}{2}} \frac{1}{s_n^2 + b_k s_n + 1} \Bigg|_{s_n = \frac{s}{\Omega_c}}$$

$$= \frac{\Omega_c}{s + \Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{s^2 + b_k \Omega_c + \Omega_c^2}$$

DESIGN PROCEDURE FOR LOWPASS DIGITAL BUTTERWORTH IIR FILTER

6. Determine the transfer function of digital filter H(z). Using the suitable transformation to transform H(s) to H(z). When the Impulse invariant transformation is employed, if T<1, then multiply H(z) by T to normalize the magnitude.

7. Realize the digital filter transfer function H(z) by a suitable structure

8. Verify the design by sketching the frequency response $H(e^{j\omega})$

$$\mathbf{H(e^{j\omega}) = H(z) / z = e^{j\omega}}$$

ASSESSMENT

1. Compare Impulse Invariant and Bilinear transformation?
2. What is Butterworth approximation?
3. How will you choose the order N for a Butterworth Filter?
4. List the Properties of Butterworth Filter.
5. Analog filter transfer function is converted to a digital filter transfer function by using either -------------------- (or) -------------------------
6. Define Sampling Time.

THANK YOU