SNS COLLEGE OF TECHNOLOGY

Coimbatore-35
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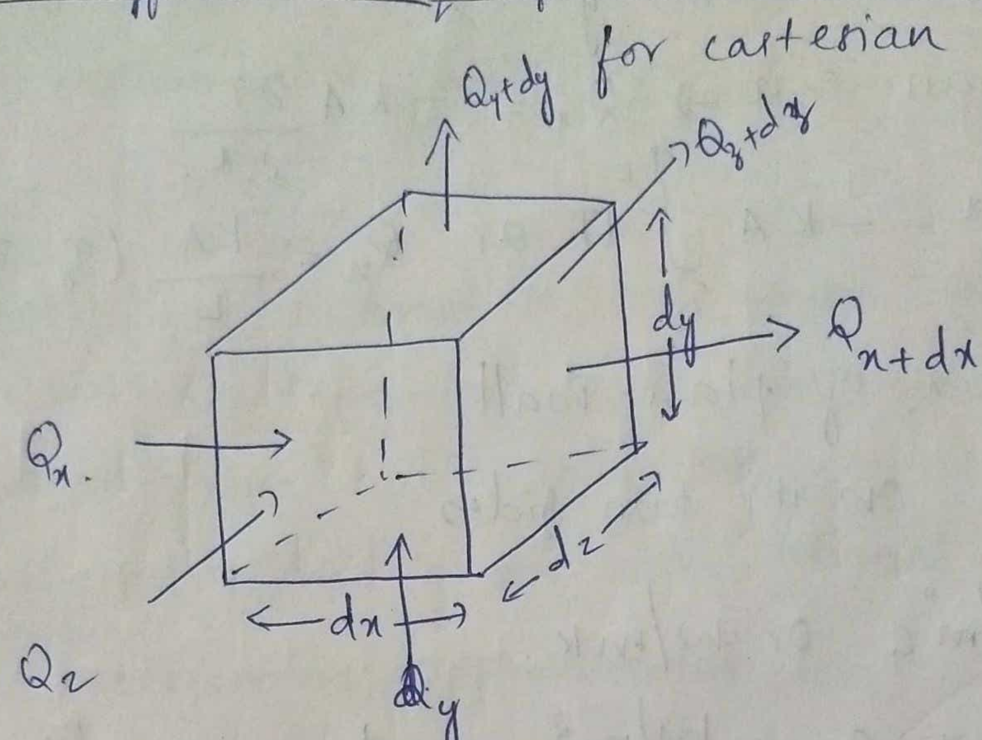
DEPARTMENT OF FOOD TECHNOLOGY

UNIT 1 – CONDUCTION

Topic 2 General differential equation of heat conduction for Cartesian and Cylindrical coordinates

General differential eqn of heat conduction for cartesian coordinates



- heat influx = Q_x.

  heat efflux = Q_{x+dx}.

  heat entry in y dire - Q_y

  " out " y " = Q_{y+dy}

  $|||^r$ for z direction.

- Let's look for energy balance eqn.

  Acc to

$$\underbrace{\left[\text{Net heat input}\right]}_{\text{part I}} + \underbrace{\left[\text{heat generated in element}\right]}_{\text{Part II}} = \underbrace{\left[\text{change in internal energy}\right]}_{\text{Part III}}$$

Part I

Net heat I/p in all direct^n = $\begin{bmatrix}\text{Heat I/p} \\ \text{in x dir}\end{bmatrix} + \begin{bmatrix}\text{Heat I/p} \\ \text{in y}\end{bmatrix} + \begin{bmatrix}\text{Heat I/p} \\ \text{in z}\end{bmatrix}$ —— ①

Net heat input = ~~input~~ entry - exit.

$$= \left[Q_x - Q_{x+dx}\right] + \left[Q_y - Q_{y+dx}\right] + \left[Q_z - Q_{z+dz}\right]$$ ← ②

Consider x direct^n

Net heat I/p in x dire = $Q_x - Q_{x+dx}$ ⟶ ③

Acc to Taylor's series.

$$Q_{x+dx} = \left[Q_x + \left[\frac{\partial}{\partial x}(Q_x) \cdot \frac{dx}{1!}\right] + \underbrace{\left[\frac{\partial^2}{\partial x^2}(Q_x + \frac{dx}{2!}\right] + \ldots}_{\text{too small} \quad \text{neglect}}\right.$$

$$Q_{x+dx} = Q_x + \frac{\partial}{\partial x}(Q_x) \cdot dx.$$

Put in eq ③

Net heat I/p in x dire $= Q_x - \left[Q_x + \frac{\partial}{\partial x}(Q_x) \cdot dx\right]$

$$= \cancel{Q_x} - \cancel{Q_x} - \frac{\partial}{\partial x}(Q_x).dx$$

$$= -\frac{\partial}{\partial x}(Q_x).dx.$$

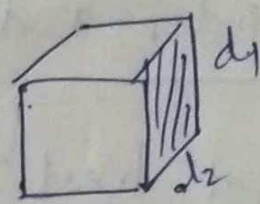
Acc to Fourier law.

$$Q_x = -K_x . A_x \frac{\partial T}{\partial x}.$$

$$= \frac{-\partial}{\partial x}\left(-K_x . A_x . \frac{\partial T}{\partial x}\right).dx.$$

A_x. – Area perpendicular to x

$= dz.dy.$



$$Q_x = -\frac{\partial}{\partial x}\left(-K_x . dy.dz . \frac{\partial T}{\partial x}\right).dx.$$

$$= K_x \, dx\,dy\,dz \frac{\partial^2 T}{\partial^2 x^2}. \longrightarrow (4)$$

Net heat I/p for y $= Q_y = k_y \, dx\,dy\,dz \frac{\partial^2 T}{\partial^2 y^2}$ — (5)

" for z $Q_z = k_z \, dx\,dy\,dz \frac{\partial^2 T}{\partial^2 z}.$ $\longrightarrow$ (6)

Put (4), (5), (6) in (2)

Net heat I/p in all dir $= \left[K_x \, dx\,dy\,dz \frac{\partial^2 T}{\partial^2 x^2}\right] + \left(K_y \, dx\,dy\,dz \frac{\partial^2 T}{\partial^2 y^2}\right.$

$$+ \left[k_z \, dx \, dy \, dz \, \frac{\partial^2 T}{\partial^2 z^2} \right]$$

Assuming isotropic mat $= k = k_x = k_y = k_z$
(same ppty)

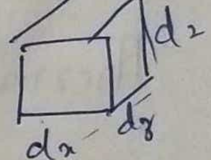
Net heat I/p in all di $= k \cdot dx \, dy \, dz \left[\frac{\partial^2 T}{\partial^2 x^2} + \frac{\partial^2 T}{\partial^2 y^2} + \frac{\partial^2 T}{\partial^2 z^2} \right]$.

$\longrightarrow$ (7).

Part II

Heat generated in the element $= \left[\text{Heat generated per unit vol} \right] \times \left[\text{Vol of eleme} \right]$

$$= \frac{kw}{m^3} \times m^3$$



$$= q_g \times dx \cdot dy \cdot dz. \longrightarrow (8).$$

Part III

change in internal energy $= m C_p \Delta T$

$$= m C_p \frac{\partial T}{\partial t}.$$

$m = \rho \cdot V$

$\frac{kg}{m^3} \times m^3$

$$= \rho \, V \, C_p \frac{\partial T}{\partial t}.$$

$$= \rho \, dx \cdot dy \cdot dz \, C_p \frac{\partial T}{\partial t} \text{ — } (9)$$

Put (7), (8), (9) in (1).

$$\left[k \cdot dx \cdot dy \cdot dz \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right]\right] + \left[q_g \cdot dx \cdot dy \cdot dz\right]$$

$$= \left[\rho \cdot dx \cdot dy \cdot dz \cdot C_p \frac{\partial T}{\partial t}\right]$$

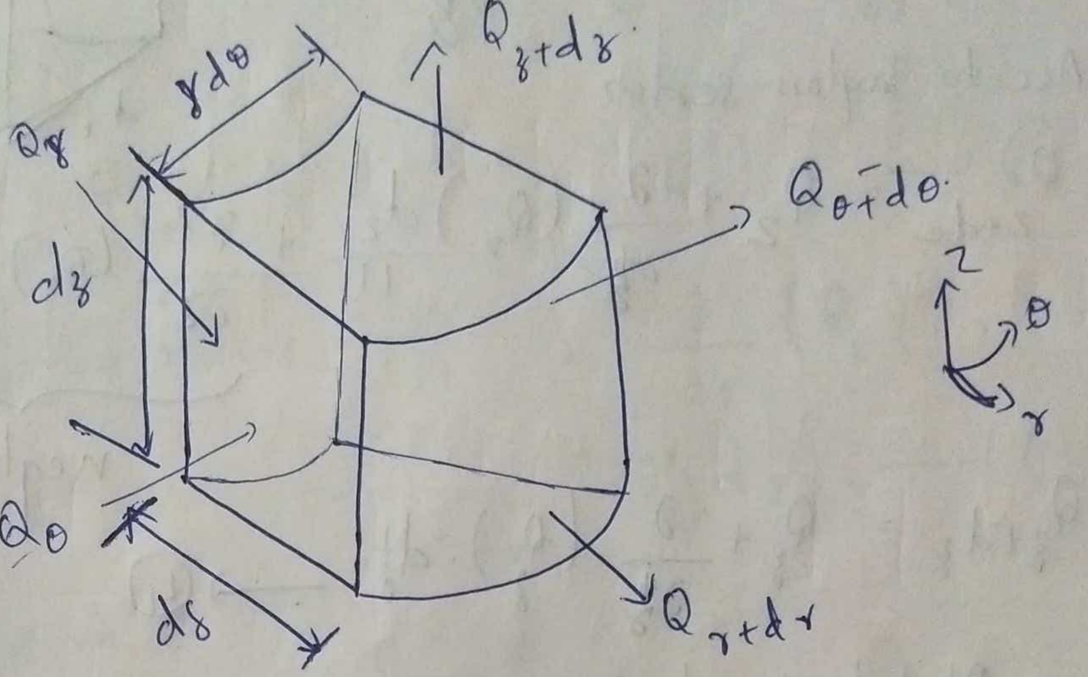
$$k \cdot \cancel{dx} \cdot \cancel{dy} \cdot dz \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q_g}{k}\right] = \left[\rho \cdot \cancel{dx} \cdot \cancel{dy} \cdot dz \cdot C_p \frac{\partial T}{\partial t}\right.$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial T^2}{\partial y^2} + \frac{\partial T^2}{\partial z^2} + \frac{q_g}{k} = \frac{\rho \cdot C_p}{k} \cdot \frac{\partial T}{\partial t}$$

Thermal diffusivity $\alpha = \frac{k}{\rho C_p}$

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial T^2}{\partial y^2} + \frac{\partial T^2}{\partial y^2} + \frac{q_g}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}}$$

General differential eqn of heat conduction for cylindrical coordinates



Energy balance eq

$$\begin{bmatrix}\text{Net heat} \\ \text{I/p}\end{bmatrix} + \begin{bmatrix}\text{Heat generated} \\ \text{in element}\end{bmatrix} = \begin{bmatrix}\text{Change in} \\ \text{IE}\end{bmatrix}$$

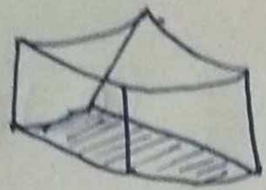
$$\underbrace{}_{\text{Part I}} \quad \underbrace{}_{\text{Part II}} \quad \underbrace{}_{\text{Part III}}$$

Net heat I/p = Net heat I/p in z dir + $\begin{bmatrix}\text{NHIP} \\ \text{in } r\end{bmatrix}$ + $\begin{bmatrix}\text{NHIP} \\ \text{in } \theta\end{bmatrix}$

Consider z direction

NHIP in z $= Q_z - Q_{z+dz}$

Acc to Fourier's law,



$$Q_z = -k(r.d\theta.dr).\frac{\partial T}{\partial z}$$

Acc to Taylors series.

$$Q_{z+dz} = Q_z + \frac{\partial}{\partial z}(Q_z)\frac{dz}{1!} + \underbrace{\frac{\partial^2}{\partial z^2}(Q_z)\frac{dz}{2!} + \cdots}_{\text{neglect.}}$$

$$Q_z + d_z = Q_z + \frac{\partial}{\partial z}(Q_z)d_z \quad — (4)$$

NHIP in z direct^n $= \cancel{Q_z} - Q_z - \frac{\partial}{\partial z}(Q_z)d_z$

$$= -\frac{\partial}{\partial z}(Q_z)d_z \quad — (5)$$

NHIP in r direct^n $= Q_r - Q_{r+dr}$ — (6)

Acc to Fourier's law, $Q_r = -k(r.d\theta.d_z).\frac{\partial T}{\partial r}$ — (7)

Acc to Taylors series

$$Q_{r+dr} = Q_r + \frac{\partial}{\partial r}(Q_r)\,d_r \quad — (8)$$

NHIP in r direct^n $= \cancel{Q_r} - \cancel{Q_r} - \frac{\partial}{\partial r}(Q_r)\,dr$

$$= -\frac{\partial}{\partial r}(Q_r)\,dr \quad — (9)$$

NHIP in θ direct^n $= Q_\theta - Q_{\theta+d\theta}$ — (10)

Acc to Fourier law, $Q_\theta = -k(d_z.d_r)\frac{\partial T}{r\,d\theta}$ — (11)

Acc to Taylor's series

$$Q_{\theta+d\theta} = Q_\theta + \frac{\partial}{r\partial\theta}(Q_\theta).r.d\theta \quad — (12)$$

NHIP in θ direct^n: $\cancel{Q_\theta} - Q_\theta - \frac{\partial}{r\partial\theta}(Q_\theta)\,d\theta.r$

$$= \frac{-\partial}{r\partial\theta}(Q_\theta)\,r.d\theta \quad — (13)$$

NHIP in all direct^n

$$= \frac{-\partial}{\partial z}(Q_z)\,dz + \frac{-\partial}{\partial r}(Q_r).dr + \frac{-\partial}{r\partial\theta}(Q_\theta)\,r.d\theta \quad \text{--- (14)}$$

Part III

Heat generated in system.

$$= q_g \times \text{Volume}$$

$$= q_g \times r.d\theta\,.dr.dz \quad \text{--- (15)}$$

Part-III

change in IE $= mc\frac{\partial T}{\partial t}$

$$= \rho \times r.d\theta.dr.dz.c.\frac{\partial T}{\partial t}$$

Put part I, II, III in (1)

$$\frac{-\partial}{\partial z}(Q_z)\,dz - \frac{\partial}{\partial r}(Q_r)\,dr - \frac{\partial}{r\partial\theta}(Q_\theta).r.d\theta) + q_g \times r.d\theta.dr.d$$

$$= \rho \times r.d\theta.dr.dz.c.\frac{\partial T}{\partial t}.$$

Put Q_z, Q_r, Q_θ in above eqn

$$-\frac{\partial}{\partial z}(-K.r.d\theta.dr)\frac{\partial T}{\partial z}.dz - \frac{\partial}{\partial r}(-K.r.d\theta.dz\frac{\partial T}{\partial r}).dr -$$

$$\frac{\partial}{r\partial\theta}(-K.dr.dz\frac{\partial T}{r\partial\theta})..r\,d\theta + q_g.r.d\theta.dr.dz)$$

$$= (\rho.r.d\theta.dr.dz.c.\frac{\partial T}{\partial t})$$

$$\left[\rho \cdot r \cdot d\theta \cdot dr \cdot d_z \, c \frac{\partial T}{\partial t}\right]$$

$$K \frac{\partial}{\partial z}\left[\left(\frac{\partial T}{\partial z}\right) r \cdot d\theta \cdot \cancel{dr} \cdot dt + K \frac{\partial}{\partial r}\left(r \cdot \frac{\partial T}{\partial r}\right) \cancel{d\theta} \, d_z \, d_r +\right.$$

$$\left. K \frac{\partial}{r \partial \theta}\left(\frac{\partial T}{r \partial \theta}\right) r \cdot d\theta \cdot dr \cdot d_z\right] + \left[q_g \cdot \cancel{r \cdot d\theta \cdot dr \cdot d_z}\right]$$

$$= \rho r \cdot \cancel{d\theta \cdot dr \, d_z} \, c \frac{\partial T}{\partial t}$$

$$K \frac{\partial^2 T}{\partial z^2} + \left[r \cdot \frac{\partial^2 T}{\partial r} + \frac{\partial T}{\partial r}\right] \frac{1}{r} + \left.\frac{\partial^2 T}{r^2 \partial^2 \theta}\right] + \frac{q_g}{K}$$

$$= \rho c \cdot \frac{\partial T}{\partial t}$$

$$\frac{\partial^2 T}{\partial z^2} + \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial T^2}{r^2 \partial^2 \theta^2} + \frac{q_g}{K}$$

$$= \frac{\rho c}{K} \frac{\partial T}{\partial t}$$

$$\frac{\partial^2 T}{\partial z^2} + \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\cancel{\partial^2} T}{\cancel{\partial r}} + \frac{\partial^2 T}{r^2 \partial^2 \theta^2} + \frac{q_g}{K}$$

$$= \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

→ u v rule.

1

$$\frac{\partial}{\partial r}\left(\underset{u}{r} \cdot \underset{v}{\frac{\partial T}{\partial r}}\right) = r \cdot \frac{\partial^2 T}{\partial r} + \frac{\partial T}{\partial r}$$