Energy density in electrostatic fields

(system)

To determine the energy present in an assembly of charges, it is necessary to calculate the amount of work to assembly them.

For three point charges, initially in an empty space.

No work is required to transfer Q_1 from infinity to P_1 because the space is initially charge free and there is no electric field. $[W=0]$

The work done in transferring Q_2 from infinity to P_2 is equal to the product of Q_2 and the potential V_{21} at P_2 due to Q_1.

Similarly work done in positioning Q_3 at P_3 is equal to $Q_3(V_{32}+V_{31})$; V_{32} & V_{31} are potentials at P_3 due to Q_2 & Q_1

Total work done in positioning 3 charges is
W_E =

$$W_E = W_1 + W_2 + W_3$$

$$= 0 + Q_2 V_{21} + Q_3(V_{31} + V_{32}) \rightarrow (1)$$

If the charges were positioned in the reverse order

$$W_E = W_3 + W_2 + W_1$$

$$= 0 + Q_2 V_{23} + Q_1(V_{12} + V_{13}) \rightarrow (2)$$

(V_1 is the potential at P_1 where Q_1 is placed; due to all other charges Q_2, Q_3.)

Adding ① & ② gives

$$2W_E = Q_1(V_{12}+V_{13}) + Q_2(V_{21}+V_{23}) + Q_3(V_{31}+V_{32})$$

$$2W_E = Q_1 V_1 + Q_2 V_2 + Q_3 (V_3)$$

$$W_E = \frac{1}{2}(Q_1 V_1 + Q_2 V_2 + Q_3 V_3)$$

If there are n point charges, Total work done is the potential energy of the s/m of charges (W_E) in Joules

Potential energy stored in the s/m of 'n' point charges

$$\boxed{W_E = \frac{1}{2}\sum_{k=1}^{n} Q_k V_k}$$

Instead of point charges, the region has continuous charge distributions,

then $W_E = \frac{1}{2}\int_l \rho_L V \, dl$ (line charge)

$W_E = \frac{1}{2}\int_s \rho_s V \, ds$ (surface charge)

$W_E = \frac{1}{2}\int_v \rho_v V \, dv$ (volume charge)

Energy stored in terms of $\vec{D}$ & $\vec{E}$:-

Consider volume charge distribution ρ_v C/m³

Since $\boxed{\rho_v = \vec{\nabla}\cdot\vec{D}}$

$$W_E = \frac{1}{2}\int_v (\vec{\nabla}\cdot\vec{D}) V \, dv$$

Vector identity (A - vector & V scalar)

$$\vec{\nabla}\cdot V\vec{A} = \vec{A}\cdot\nabla V + V(\vec{\nabla}\cdot\vec{A})$$

$$(\vec{\nabla}\cdot\vec{A})V = \vec{\nabla}\cdot V\vec{A} - \vec{A}\cdot\nabla V$$

Applying, we get

$$W_E = \frac{1}{2}\int_v \vec{\nabla}\cdot V\vec{D} \, dv - \frac{1}{2}\int_v \vec{D}\cdot\nabla V \, dv$$

Appling div theorem for the first term

$$W_E = \frac{1}{2}\oint_S V\vec{D}\cdot d\vec{s} - \frac{1}{2}\int_V \vec{D}\cdot\nabla V\, dv$$

V varies as $1/r^2$ and D as $1/r^3$ for dipole.

VD in the first term must vary at least as $1/r^3$ while ds varies as r^2. Hence total integral varies as $1/r$.

If the surface S becomes large, $r \to \infty$, $1/r \to 0$, It becomes zero.

Hence closed surface integral is zero.

$$W_E = -\frac{1}{2}\int_V \vec{D}\cdot\vec{\nabla}V\, dv = -\frac{1}{2}\int_V D(-E)\, dv$$

Since $\vec{E} = -\nabla V$

$$\boxed{W_E = \frac{1}{2}\int_V \bar{D}\cdot\bar{E}\, dv} \quad J$$

$$W_E = \frac{1}{2}\int_V E\cdot\varepsilon_0 \bar{E}\, dv \qquad E\cdot E = E^2$$

$\bar{D} = \varepsilon_0\vec{E}$

$$\boxed{W_E = \frac{1}{2}\int_V \varepsilon_0 E^2\, dv} \quad \text{or} \quad \boxed{W_E = \frac{1}{2}\int_V \frac{D^2}{\varepsilon_0}\, dv} \quad J$$

Electrostatic energy density W_E (in J/m^3)

$$W_E = \frac{dW_E}{dv} = \frac{d\int \frac{1}{2} D\cdot E\, dv}{dv}$$

$$\boxed{W_E = \frac{1}{2} D\cdot E}$$

or

$$\boxed{W_E = \frac{1}{2}\varepsilon_0 E^2 = \frac{D^2}{2\varepsilon_0}}$$

Integrated over the volume, for the total energy present.