# Unit I - Static Electric Field

The static electric field could be termed as Electrostatic.
An electrostatic field is produced by a static charge distribution.

A typical example of such a field is found in a cathode-ray tube.

Applications of electrostatics:

- Electric power transmission
- X-ray machines
- lightning protection.

**Vector Algebra** :- Objective : To know the mathematical tool that helps to realize the Electromagnetic concepts

- Vector analysis is mathematical tool with which electromagnetic concepts are most conveniently expressed.
- A quantity can be either a scalar or a vector. A scalar is a quantity that has only magnitude

  Eg: Time, mass, distance, temperature, electric potential.

- A vector is a quantity that has both magnitude and direction.

  Eg: Velocity, force, displacement electric field intensity.

- A field is a function that specifies a particular quantity everywhere in the region.

  Example for scalar fields ⇒ electric potential in a region.

Vector field ⇒ Gravitational force on a body in Space & velocity of raindrops in the atmosphere.

## Unit Vector :-

A unit vector along $\vec{A}$ is defined as a vector whose magnitude is unity, and its direction is along A.

$$\hat{a} = \frac{\vec{A}}{|\vec{A}|}$$

or $\vec{A} = |A|\,\hat{a}$

A vector in Cartesian (rectangular) coordinates may be represented as

$$\vec{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z \quad \text{or} \quad (A_x, A_y, A_z)$$

$A_x, A_y, A_z$ are called components of $\vec{A}$ in the x, y, z coordinates. $a_x, a_y, a_z$ are unit vectors.

Magnitude of the vector A is given by

$$|A| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

and the unit vector along A is given by

$$\hat{a} = \frac{A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z}{|A|}$$

# Vector addition and subtraction

- vector addition is :

$$\vec{C} = \vec{A} + \vec{B}$$

If $\vec{A} = (A_x, A_y, A_z)$ and $B = (B_x, B_y, B_z)$

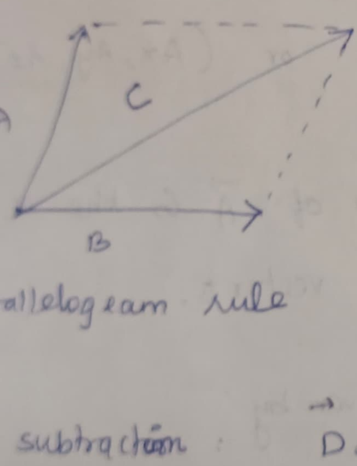
$$\vec{C} = (A_x + B_x)\hat{a}_x + (A_y + B_y)\hat{b}_y + (A_z + B_z)\hat{a}_z$$

- Vector subtraction is

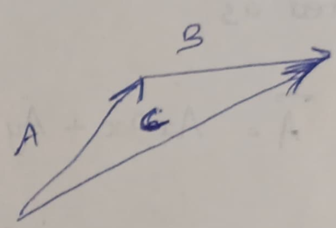
$$\vec{D} = \vec{A} - \vec{B}$$

$$\vec{D} = (A_x - B_x)\hat{a}_x + (A_y - B_y)\hat{a}_y + (A_z - B_z)\hat{a}_z$$

Vector addition. $\vec{C} = \vec{A} + \vec{B}$

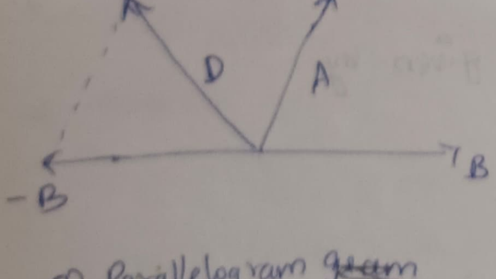



a) Parallelogram rule

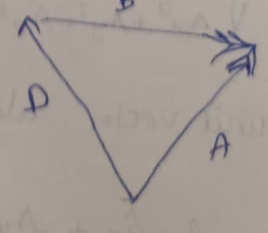



b) head-to-tail rule

vector subtraction : $\vec{D} = \vec{A} - \vec{B}$




a) Parallelogram ~~gram~~



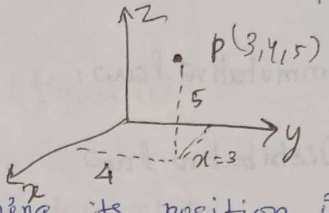

b) head-to-tail rule

# Position and Distance vectors

A point P in cartesian coordinates may be represented by $(x, y, z)$.

- A position vector $r_p$ (or radius vector) of point P is defined as the directed distance from the origin O to P.

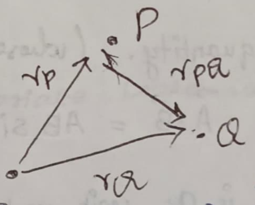
$$\vec{r}_p = \vec{OP} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$$




- A position vector is useful in defining its position in space.

- A distance vector is the displacement from one point to another.

If two points P and Q are given by $(x_1, y_1, z_1)$ $(x_P, y_P, z_P)$ and $(x_2, y_2, z_2)$ $(x_Q, y_Q, z_Q)$ the distance vector (or separation vector) is the displacement from P to Q.




$$\vec{r}_{PQ} = r_Q - r_P$$

$$= (x_Q - x_P)\hat{a}_x + (y_Q - y_P)\hat{a}_y + (z_Q - z_P)\hat{a}_z$$

$$= (x_2 - x_1)\hat{a}_x + (y_2 - y_1)\hat{a}_y + (z_2 - z_1)\hat{a}_z$$

## Vector Multiplication:

1. Scalar (or) Dot Product $A \cdot B$
2. Vector (or) Cross Product $A \times B$
3. Scalar triple product $A \cdot (B \times C)$
4. Vector triple Product $A \times (B \times C)$

Scalar Product

**Dot Product:** The dot product of two vectors A & B,

$$\vec{A}\cdot\vec{B} = AB\cos\theta_{AB}$$

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

A and B are said to be orthogonal (or) perpendicular with each other if $\vec{A}\cdot\vec{B} = 0$.

- Commutative law: $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$
- Distributive law: $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$
- $\vec{A}\cdot\vec{A} = |A|^2 = A^2$

$$a_x\cdot a_y = a_y\cdot a_z = a_z\cdot a_x = 0$$

$$a_x\cdot a_x = a_y\cdot a_y = a_z\cdot a_z = 1$$

**Cross Product:** Vector Product

Cross product of two vectors A and B, $A\times B$ is a vector quantity. (whose magnitude in the area of the Parallelogram) & direction of advance of a right handed screw)

$$\vec{A}\times\vec{B} = AB\sin\theta_{AB}\cdot a_n$$

figure(i)

$a_n$ is a unit vector normal to the plane containing A and B.

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

- It is not commutative: $\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$
- It is anticommutative: $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$
- It is not associative: ~~Ax~~

$$\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C}$$

- It is distributive:

$$A \times (B + C) = A \times B + A \times C$$

- $A \times A = 0$

$a_x \times a_y = a_z$ $\quad$ $a_y \times a_x = -a_z$ $\quad$ $a_x \times a_x = 0$

$a_y \times a_z = a_x$ $\quad$ $a_z \times a_y = -a_x$

$a_z \times a_x = a_y$ $\quad$ $a_x \times a_z = -a_y$ (figure ii)

## Coordinate Systems and Transformation

Objectives: Importance of coordinate system for the analysis of Electromagnetic fields.

Why coordinate systems?

Physical quantities dealing with Electromagnetics are functions of Space and time. In order to describe the spatial variations of the quantities, we must be able to define all the points uniquely in space.

This requires an appropriate coordinate system.

It may be orthogonal or nonorthogonal.

An orthogonal system is one in which the coordinates are mutually perpendicular.

Examples of Orthogonal C.S:

1. Cartesian (or rectangular)
2. Circular Cylindrical
3. Spherical
4. Elliptic Cylindrical.

Cylindrical etc.,

## Cartesian Coordinates $(x, y, z)$

A point P can be represented as $(x, y, z)$.

The ranges of the variables are

$$-\infty < x < \infty$$
$$-\infty < y < \infty$$
$$-\infty < z < \infty$$

A vector in cartesian is written as

$(A_x, A_y, A_z)$ or $\boxed{A_x \hat{a_x} + A_y \hat{a_y} + A_z \hat{a_z}}$

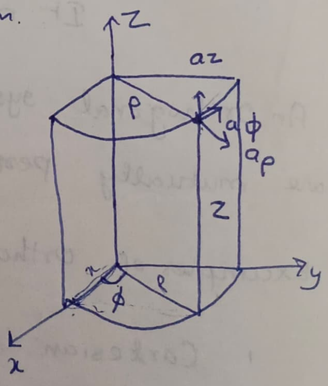
## Circular Cylindrical Coordinate $(\rho, \phi, z)$

A point P is represented as $(\rho, \phi, z)$. $\rho$ is the radius of the cylinder passing through P or the radial distance from the z-axis.

$\phi$ is the azimuthal angle, measured from the x axis in the xy plane.

z is same as in the cartesian system.

$$0 \le \rho \le \infty$$
$$0 \le \phi \le 2\pi$$
$$-\infty \le z < \infty$$




A vector A in cylindrical system

$(A_\rho, A_\phi, A_z)$ or $\boxed{A_\rho \hat{a_\rho} + A_\phi \hat{a_\phi} + A_z \hat{a_z}}$

$a_\rho, a_\phi, a_z$ are unit vectors in the $\rho, \phi, z$ directions.

$a_\rho \cdot a_\rho = a_\phi \cdot a_\phi = a_z \cdot a_z = 1$

$a_\rho \cdot a_\phi = a_\phi \cdot a_z = a_z \cdot a_\rho = 0$

$a_\rho \times a_\phi = a_z$

$a_\phi \times a_z = a_\rho$

$a_z \times a_\rho = a_\phi$

Relationship between the variables $(x, y, z)$ of the Cartesian & $(\rho, \phi, z)$ of Cylindrical.

$\rho = \sqrt{x^2 + y^2}$ $\qquad x = \rho \cos\phi$

$\phi = \tan^{-1}(y/x)$ $\qquad y = \rho \sin\phi$

$z = z$ $\qquad z = z$

Relationship between $(A_x, A_y, A_z)$ and $(A_\rho, A_\phi, A_z)$

$A_\rho = A \cdot a_\rho$

$A_\rho = A_x \cos\phi + A_y \sin\phi$

$A_\phi = A \cdot a_\phi$

$A_\phi = -A_x \sin\phi + A_y \cos\phi$

$A_z = A \cdot a_z$

$A_z = A_z$

| | $a_\rho$ | $a_\phi$ | $a_z$ |
|---|---|---|---|
| $a_x$ | $\cos\phi$ | $-\sin\phi$ | 0 |
| $a_y$ | $\sin\phi$ | $\cos\phi$ | 0 |
| $a_z$ | 0 | 0 | 1 |

In matrix form

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$A_\rho = \vec{A} \cdot \hat{a}_\rho = (A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z) \cdot \hat{a}_\rho$

$= A_x \hat{a}_x \cdot \hat{a}_\rho + A_y \hat{a}_y \cdot \hat{a}_\rho + A_z \hat{a}_z \cdot \hat{a}_\rho$

$= A_x \cos\phi + A_y \sin\phi + 0$

$(A_\rho, A_\phi, A_z) \longrightarrow (A_x, A_y, A_z)$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ +\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

## Spherical coordinates $(r, \theta, \phi)$

Point P is represented as $(r, \theta, \phi)$

$r$ is the distance from the origin to Point P or the radius of a sphere.

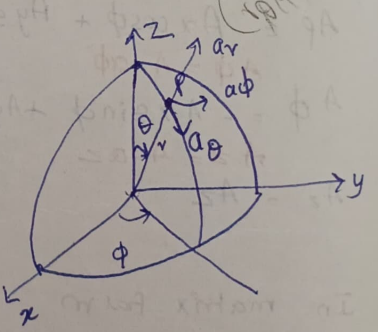
$\theta$ (called as colatitude) is the angle between Z axis and position vector of P.

$\phi$ is measured from the $x$ axis (same azimuthal angle in cylindrical coordinates)

$0 \le r \le \infty$

$0 \le \theta \le \pi$

$0 \le \phi < 2\pi$




A vector A in spherical

$(A_r, A_\theta, A_\phi)$ or $\boxed{A_r \hat{a_r} + A_\theta \hat{a_\theta} + A_\phi \hat{a_\phi}}$

$a_r \cdot a_r = a_\theta \cdot a_\theta = a_\phi \cdot a_\phi = 1$

$a_r \cdot a_\theta = a_\theta \cdot a_\phi = a_\phi \cdot a_r = 0$

$a_r \times a_\theta = a_\phi$

$a_\theta \times a_\phi = a_r$

$a_\phi \times a_r = a_\theta$

Relationship between cartesian and spherical.

$r = \sqrt{x^2+y^2+z^2}$

$x = r\sin\theta\cos\phi$

$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}$ (or) $\cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$ (r)

$y = r\sin\theta\sin\phi$

$\phi = \tan^{-1}(y/x)$

$z = r\cos\theta$

$A_r = \vec{A}\cdot\hat{a_r}$ $\quad A_\phi = \vec{A}\cdot\hat{a_\phi}$ $\quad A_z = \vec{A}\cdot\hat{a_z}$

$A_r = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$

$A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$

$A_\phi = -A_x\sin\phi + A_y\cos\phi$

In matrix form

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Inverse transformation

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

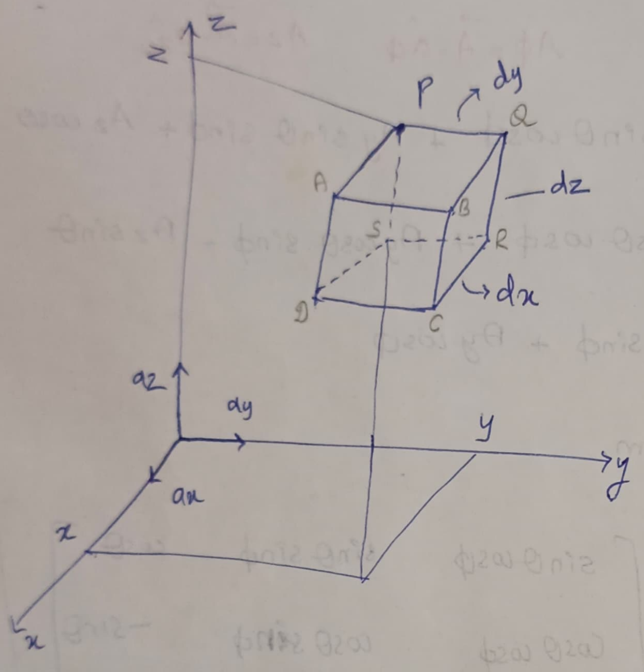
| $\cdot$ | $a_r$ | $a_\theta$ | $a_\phi$ |
|---|---|---|---|
| $a_x$ | $\sin\theta\cos\phi$ | $\cos\theta\cos\phi$ | $-\sin\phi$ |
| $a_y$ | $\sin\theta\sin\phi$ | $\cos\theta\sin\phi$ | $\cos\phi$ |
| $a_z$ | $\cos\theta$ | $-\sin\theta$ | $0$ |

# Vector Calculus:

Objective: To learn the mathematical techniques and their application in Electromagnetics

## Differential length, Area & Volume:

### a) Cartesian coordinates:




Differential elements in the cartesian coordinate system

* Differential displacement is given by

$$\vec{dl} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$$

* Differential normal area is given by

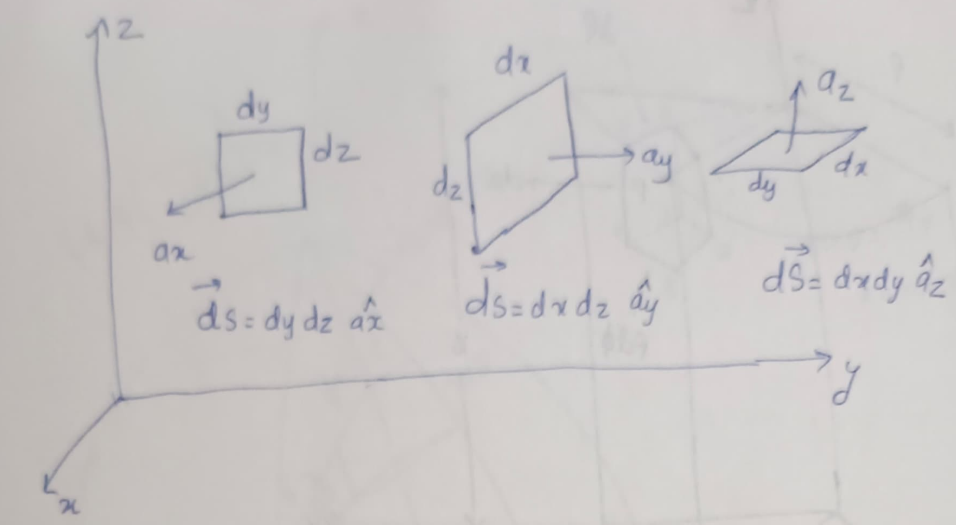
$$\vec{ds} = dy\,dz\,\hat{a}_x$$
$$dx\,dz\,\hat{a}_y$$
$$dx\,dy\,\hat{a}_z$$

* Differential volume

$$dv = dx\,dy\,dz$$

not a vector

## Differential normal areas in Cartesian coordinates




Note:

- $dl$ & $ds$ are vectors.
  $dv$ is a scalar.

- When moving along P to Q, $dl = dy\, a_y$.

- Differential surface (area) element ds

$$\vec{ds} = ds\,\hat{a_n}$$

where ds is the area of the surface element and $a_n$ is a unit vector normal to the surface ds.

- For example $\vec{ds} = dy\,dz\,\hat{a_x}$ for surface ABCD
  $\vec{ds} = -dy\,dz\,\hat{a_x}$ For PQRS because $\hat{a_n} = -\hat{a_x}$

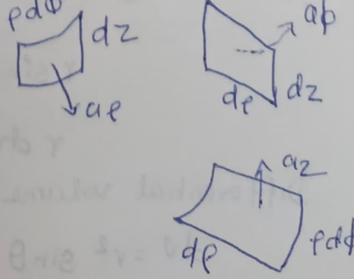
## b) Cylindrical coordinates

* differential displacement

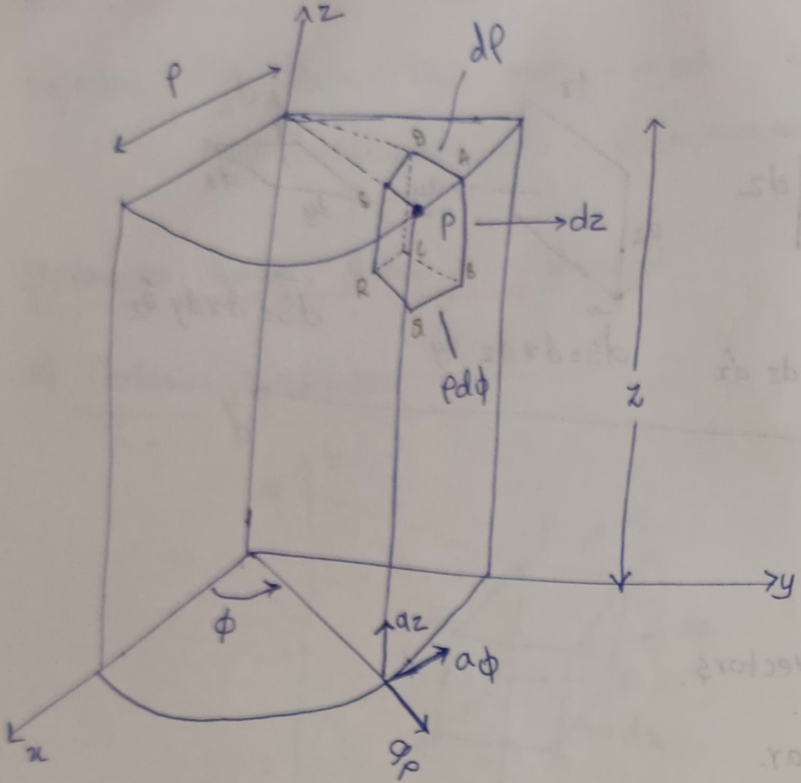
$$\vec{dl} = d\rho\,\hat{a_\rho} + \rho\,d\phi\,\hat{a_\phi} + dz\,\hat{a_z}$$

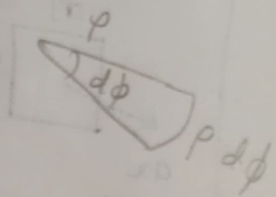
* Differential normal area

$$\vec{ds} = \rho\,d\phi\,dz\,\hat{a_\rho}$$
$$d\rho\,dz\,\hat{a_\phi}$$
$$\rho\,d\phi\,d\rho\,\hat{a_z}$$




Diff. normal areas in cylindrical coordinates




Differential elements in cylindrical coordinates

* Differential volume

$$dv = \rho \, d\rho \, d\phi \, dz$$

c) Spherical coordinates $\quad dr, \; r\,d\theta, \; r\sin\theta\, d\phi$

* Differential displacement

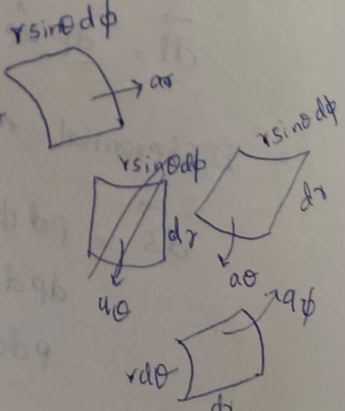
$$\vec{dl} = dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\, d\phi\,\hat{a}_\phi$$

* Differential normal area

$$ds = r^2 \sin\theta \, d\theta \, d\phi \, \hat{a}_r$$

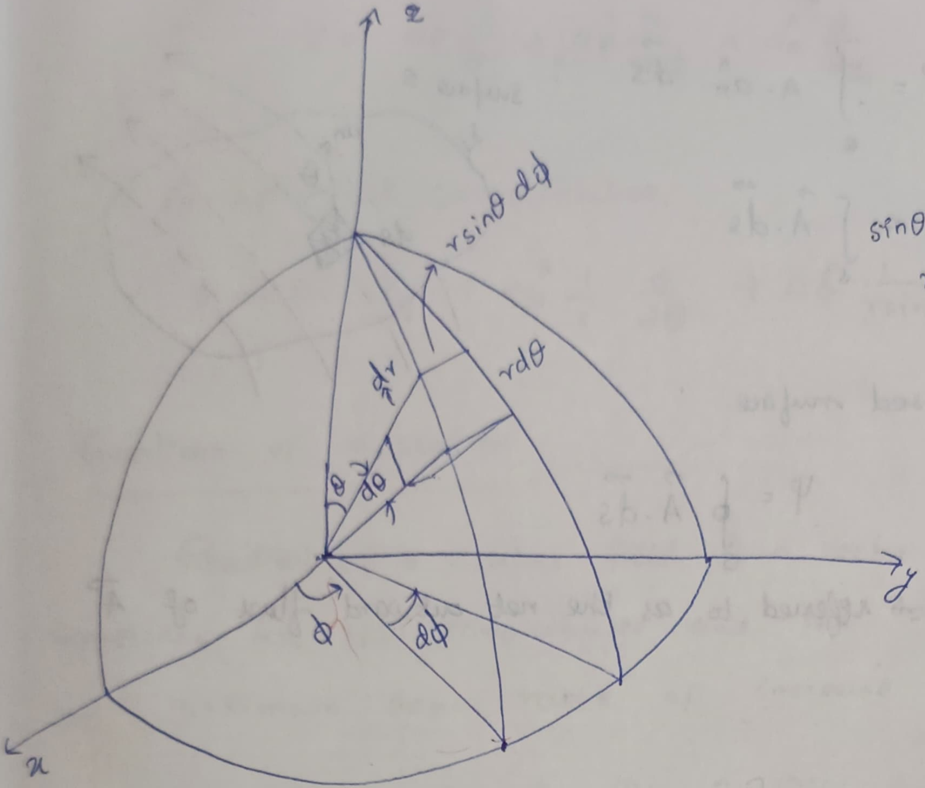
$$r\sin\theta \, dr\, d\phi \, \hat{a}_\theta$$

$$r\, dr\, d\theta \, \hat{a}_\phi$$




* Differential volume

$$dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$$




$\sin\theta = x/r$

$x = r\sin\theta$

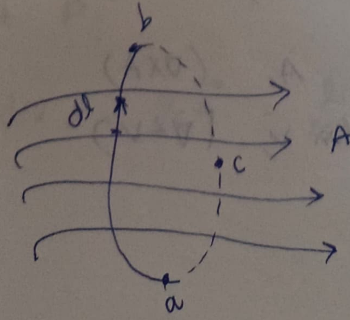
Line, Surface and Volume integrals :-

1. The line integral $\int_L \vec{A} \cdot d\vec{l}$ is the integral of the tangential component of A along curve L.

2. If the path of integration is a closed curve such as abca, becomes a closed contour integral

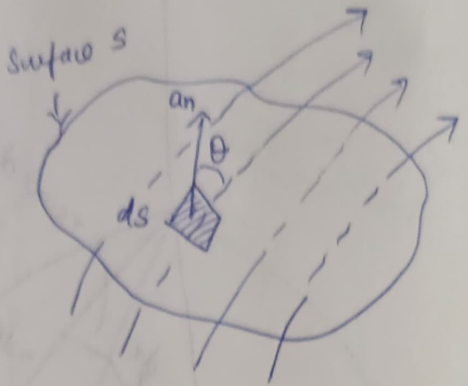
$$\oint_L \vec{A} \cdot d\vec{l}$$

which is called the circulation of A around L.



3. Surface integral or flux of $\vec{A}$ through S

$$\Psi = \int_S A \cdot \hat{a}_n \, \vec{ds}$$




or

$$\Psi = \int_S \vec{A} \cdot \vec{ds}$$

For a closed surface

$$\Psi = \oint_S \vec{A} \cdot \vec{ds}$$

which is referred to as the net outward flux of $\vec{A}$ from S.

4. The integral $\int_v \rho_v \, dv$ is the volume integral of the scalar $\rho_v$ over the volume $v$.

## Vector differential Operator:

(or) The del operator ($\nabla$)

In cartesian coordinates, $\nabla = \frac{\partial}{\partial x} \hat{a_x} + \frac{\partial}{\partial y} \hat{a_y} + \frac{\partial}{\partial z} \hat{a_z}$

Also known as the gradient operator.

It is used to define

1. The gradient of a scalar V, ($\nabla V$)
2. The divergence of a vector A, ($\nabla \cdot A$)
3. The curl of a vector A, ($\nabla \times A$)
4. The laplacian of a scalar V, ($\nabla^2 V$)

$\nabla$ in ~~cartesian~~ cylindrical coordinates,

$$\nabla = \hat{a}_\rho \frac{\partial}{\partial \rho} + \frac{1}{\rho}\hat{a}_\phi \frac{\partial}{\partial \phi} + \hat{a}_z \frac{\partial}{\partial z}$$

$\nabla$ in spherical coordinates,

$$\nabla = \hat{a}_r \frac{\partial}{\partial r} + \hat{a}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{a}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

## Gradient of a scalar

Gradient of a scalar field is a vector that represents both the magnitude and the direction of the maximum space rate of increase of V.

$$\text{grad } V = \frac{\partial V}{\partial x}\hat{a_x} + \frac{\partial V}{\partial y}\hat{a_y} + \frac{\partial V}{\partial z}\hat{a_z}$$

(or $\nabla V$)

In cylindrical coordinates,

$$\nabla V = \frac{\partial V}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a_\phi} + \frac{\partial V}{\partial z}\hat{a_z}$$

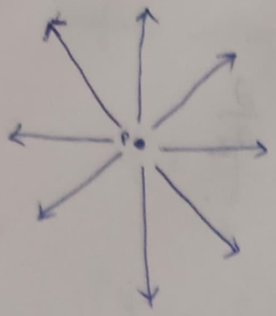
In spherical coordinates

$$\nabla V = \frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi$$

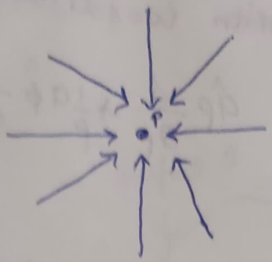
## Divergence of a vector.

The divergence of A at a given point P is the outward flux per unit volume as the volume shrinks about P.

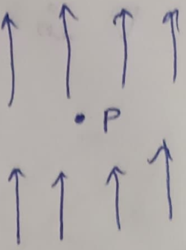
$$\text{div } A = \nabla \cdot \vec{A} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{A} \cdot \vec{ds}}{\Delta v}$$



Positive divergence



Negative divergence




zero divergence

physically, divergence of a vector field is a measure of how much the field diverges or emanates from that point.

or simply it is the limit of the field's source strength per unit volume (or source density).

In cartessian system $\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

In cylindrical $\nabla \cdot \vec{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$

In spherical $\nabla \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (A_\theta \sin\theta) + \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial \phi}$

Properties:

* Divergence of a vector field is a scalar.
* $\nabla \cdot (A + B) = \nabla \cdot A + \nabla \cdot B$
* $\nabla \cdot (VA) = V \nabla \cdot A + A \cdot \nabla V$

## Divergence theorem :

The divergence theorem states that the total outward flux of a vector field A through the closed surface S is the same as the volume integral of the divergence of A.

$$\oint_S \vec{A} \cdot d\vec{s} = \int_V \vec{\nabla} \cdot \vec{A} \, dv$$

**Proof :**

WKT $\nabla \cdot A = \dfrac{\partial Ax}{\partial x} + \dfrac{\partial Ay}{\partial y} + \dfrac{\partial Az}{\partial z}$

Take volume integral on both the sides

$$\iiint_V \nabla \cdot A \, dv = \iiint_V \left( \frac{\partial Ax}{\partial x} + \frac{\partial Ay}{\partial y} + \frac{\partial Az}{\partial z} \right) dx\,dy\,dz$$

Consider element volume in x direction

$$\iiint_V \frac{\partial Ax}{\partial x} \, dx\,dy\,dz = \iint \left\{ \int \frac{\partial Ax}{\partial x} dx \right\} dy\,dz$$

$$= \iint Ax \, dy\,dz$$

$$= \iint Ax \, ds_x$$

$$\iiint_V \frac{\partial Ay}{\partial y} \, dx\,dy\,dz = \iint_S Ay \, ds_y$$

$$\iiint_V \frac{\partial Az}{\partial z} \, dx\,dy\,dz = \iint_S Az \, ds_z$$

$$\therefore \iiint \nabla \cdot A \, dv = \iint_S A_x \, ds_x + \iint A_y \, ds_y + \iint_S A_z \, ds_z$$

$$= \oiint_S A \cdot ds$$

$$\therefore \iiint \nabla \cdot A \, dv = \iint A \cdot ds$$

or $$\int_V \vec{\nabla} \cdot \vec{A} \, dv = \oint_S \vec{A} \cdot \vec{ds}$$

# Curl of a vector:

Let Circulation of a vector field A around a closed path L as $\oint_L \vec{A} \cdot d\vec{l}$.

The curl of $\vec{A}$ is an axial or rotational vector whose magnitude is the maximum circulation of $\vec{A}$ per unit area as the area tends to zero & whose direction is the normal direction of the area when area is oriented to make the circulation maximum.

$$\text{Curl } \vec{A} = \vec{\nabla} \times \vec{A} = \left( \lim_{\Delta s \to 0} \frac{\oint_L A \cdot dl}{\Delta s} \right)_{max} \hat{a}_n$$

where area $\Delta s$ is bounded by the curve L

$\hat{a}_n$ is the unit vector normal to the surface $\Delta S$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{a_x} & \hat{a_y} & \hat{a_z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

In cylindrical, $$\vec{\nabla} \times \vec{A} = \frac{1}{\rho} \begin{vmatrix} \hat{a_\rho} & \rho\hat{a_\phi} & \hat{a_z} \\ \partial/\partial \rho & \partial/\partial \phi & \partial/\partial z \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

In spherical, $$\vec{\nabla} \times \vec{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{a_r} & r\hat{a_\theta} & r\sin\theta\, \hat{a_\phi} \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial \phi \\ A_r & rA_\theta & r\sin\theta\, A_\phi \end{vmatrix}$$

## Properties of curl

* The curl of a vector field is another vector field.

* The divergence of the curl of a vector field vanishes.

i.e. $\nabla \cdot (\nabla \times A) = 0$

* The curl of the gradient of a scalar field vanishes.

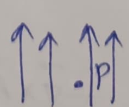
i.e. $\nabla \times \nabla V = 0$

## Physical significance:

Curl provides the maximum value of the circulation of the field per unit area (or circulation density).

Indicates the direction along which this maximum value occurs.


Curl at P point out of the page




Curl at P is zero

## Stokes's theorem:

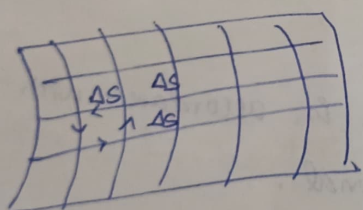
States that the circulation of a vector field $\vec{A}$ around a (closed) path L is equal to the surface integral of the curl of $\vec{A}$ over the open surface S bounded by L.

$$\oint_L \vec{A}\cdot d\vec{l} = \int_S (\vec{\nabla}\times\vec{A})\cdot \vec{ds}$$

**Proof:**

Consider any surface with area S. It is subdivided into different areas $\Delta s$. By ampere's law,

$$\oint \vec{A}\cdot\vec{dl} = \oint_1 A\cdot dl + \oint_2 A\cdot dl + \dots \rightarrow ①$$




from the definition of curl.

$$\underset{\Delta s\to 0}{Lt} \frac{\oint_L \vec{A}\cdot\vec{dl}}{\Delta s}\,\hat{a}_n = \nabla\times A$$

~~sub ② in ①~~

$$\oint \vec{A}\cdot\vec{dl} = \underset{\Delta s\to 0}{Lt} (\nabla\times A)\cdot\Delta s$$

sub ② in ①

$$\oint \vec{A}\cdot d\vec{l} = \lim_{\Delta s_1 \to 0} (\nabla \times A)\cdot \Delta s_1 + \lim_{\Delta s_2 \to 0} (\nabla \times A)\cdot \Delta s_2 + \lim_{\Delta s_3 \to 0} (\nabla \times A)\cdot \Delta s_3$$

$$\oint \vec{A}\cdot d\vec{l} = \oint_S (\nabla \times A)\cdot ds$$

Solenoidal & Irrotational

* A divergence less field is called as solenoidal field.

$\nabla \cdot \vec{A} = 0$

Eg: Magnetic field

$\nabla \cdot B = 0$ means the magnetic flux lines close upon themselves and that there are no magnetic sources or sinks ($B = \nabla \times A$)

* A curl-free vector field is called as an irrotational or conservative field.

Eg. Electrostatic field.

$\nabla \times E = 0$

We may classify vector fields in accordance with their being solenoidal and/or irrotational.

1. Solenoidal & irrotational if $\nabla \cdot F = 0$ and $\nabla \times F = 0$

Eg: Static electric field in a charge free region

2. Solenoidal but not irrotational if $\nabla \cdot F = 0$, $\nabla \times F \neq 0$

Eg: A steady magnetic field.

3. Irrotational but not solenoidal if

$\nabla \times F = 0$ and $\nabla \cdot F \neq 0$

Eg: Static electric field in a charged region

4. Neither solenoidal nor irrotational if

$\nabla \cdot F \neq 0$ and $\nabla \times F \neq 0$

Eg: An electric field in a charge medium with a time-varying magnetic field

① Proof of $\nabla\cdot(\nabla\times A) = 0$

For any vector field $\vec{A}$, show explicitly that $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$ i.e, the divergence of the curl of any vector field is zero.

Solution :

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{a_x} & \hat{a_y} & \hat{a_z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

$$\vec{\nabla}\times\vec{A} = \hat{a_x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{a_y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{a_z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = \left(\frac{\partial}{\partial x}\hat{a_x} + \frac{\partial}{\partial y}\hat{a_y} + \frac{\partial}{\partial z}\hat{a_z}\right)\cdot(\vec{\nabla}\times\vec{A})$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z} - \frac{\partial^2 A_z}{\partial x \partial y} + \frac{\partial^2 A_x}{\partial y \partial z} + \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y}$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$$

(2) Prove $\vec{\nabla} \times \vec{\nabla} V = 0$

Show that Curl of the gradient of any scalar field vanishes.

Solution:

$$\text{grad } V = \frac{\partial V}{\partial x} \hat{a_x} + \frac{\partial V}{\partial y} \hat{a_y} + \frac{\partial V}{\partial z} \hat{a_z}$$

$$\text{Curl} \cdot \text{grad } V = \begin{vmatrix} \hat{a_x} & \hat{a_y} & \hat{a_z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{vmatrix}$$

$$= \hat{a_x}\left[\frac{\partial^2 V}{\partial y \partial z} - \frac{\partial^2 V}{\partial y \partial z}\right] - \hat{a_y}\left[\frac{\partial^2 V}{\partial x \partial z} - \frac{\partial^2 V}{\partial x \partial z}\right] +$$

$$\hat{a_z}\left[\frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial x \partial y}\right]$$

$$\boxed{(\vec{\nabla} \times \vec{\nabla} V) = 0}$$

* If a vector field is curl-free, then it can be expressed as the gradient of a scalar field.

Eg: Let a vector field E. If $\nabla \times E = 0$, then $E = -\nabla V$.