Forward Kinematics

"Finding the end effector given the joint angles"

Types of robot joints

- Rotary
  - Angle θ about **z** axis
- Prismatic/sliding
  - Linear displacement *d* along axis of joint **z**
- Hooke (2-D Rotary)
  - 2-D of freedom described by Yaw and Pitch
- Spherical
  - Described by 3 rotation axes (wrist & shoulder)

Robot configurations

- Cartesian(inspection)
  - 3 Prismatic joints
- SCARA (pick and place)
  - Min 3 axes (Z axis and 2+ rotary joints)
- Spherical Wrist
  - Most common type 6-D freedom
  - hooke shoulder, rotary elbow and roll-pitch-roll wrist
- …

Setting up the coordinate system: Denavit-Hartenberg Coordinates

- Key basis for DH coordinates
  - There is only one normal between two lines in space
  - Except: Parallel lines and intersecting lines (common normal has zero length)
  - Suppose there are two joint axes $\mathbf{z}_i$ and $\mathbf{z}_{i-1}$

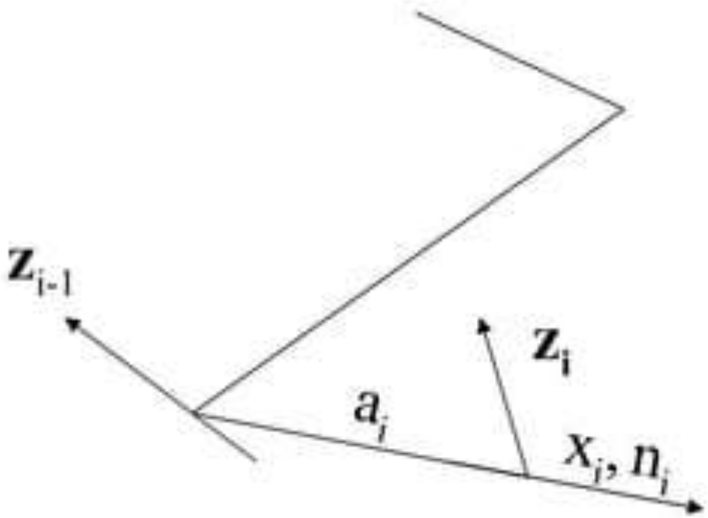
$$n_i = \frac{\mathbf{z}_{i-1} \times \mathbf{z}_i}{\left\| \mathbf{z}_{i-1} \times \mathbf{z}_i \right\|}$$

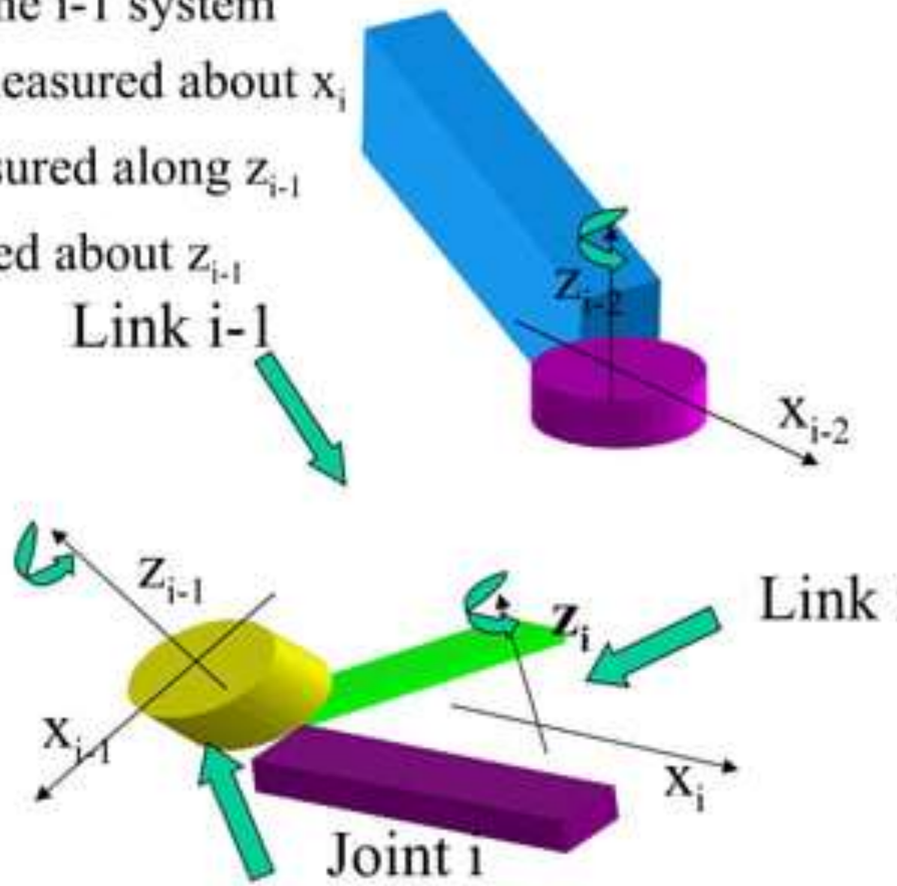
- DH coordinates are numbered relative to the base 0
- Joint i connects link i-1 to link i
- The intersection of n_i with z_i defines the origin of the link
- We take z_i to be parallel to the n_i
- The coordinate system for link i is at the distal end
- The coordinate system at joint i is the i-1 system

∀ α_i is the skew angle from z_{i-1} to z_i measured about x_i

- d_i is the distance from x_{i-1} to x_i measured along z_{i-1}

∀ θ_i is the angle from x_{i-1} to x_i measured about z_{i-1}



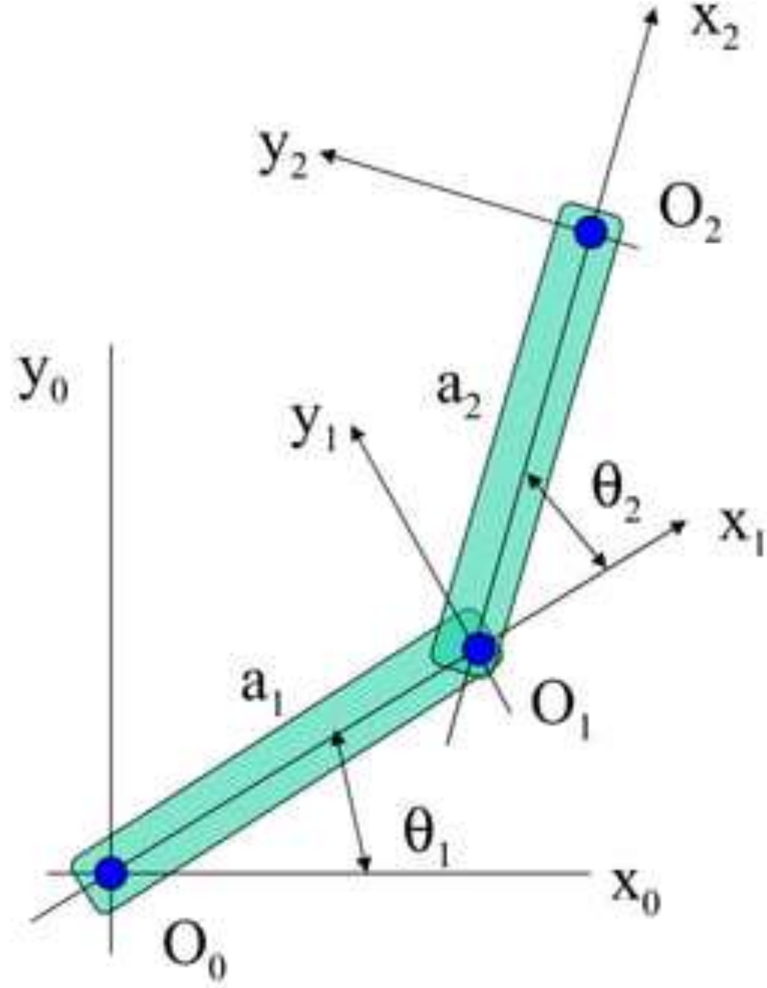


Parallel or intersecting axes

- When neighbouring axes intersect $a_i = 0 \Rightarrow |n_i| = 0$
  - Arbitrary choice $x_i \,||\, n_i$ or $-n_i$
- For Parallel axes
  - Chose x_i that intersects x_{i-1} at O_{i-1} (prefered)
  - Chose x_i that intersects x_{i+1} at O_{i+1}
- In these cases DH parameters are very sensitive to small changes in alignment
  - An alternative is the Hayati coordinates

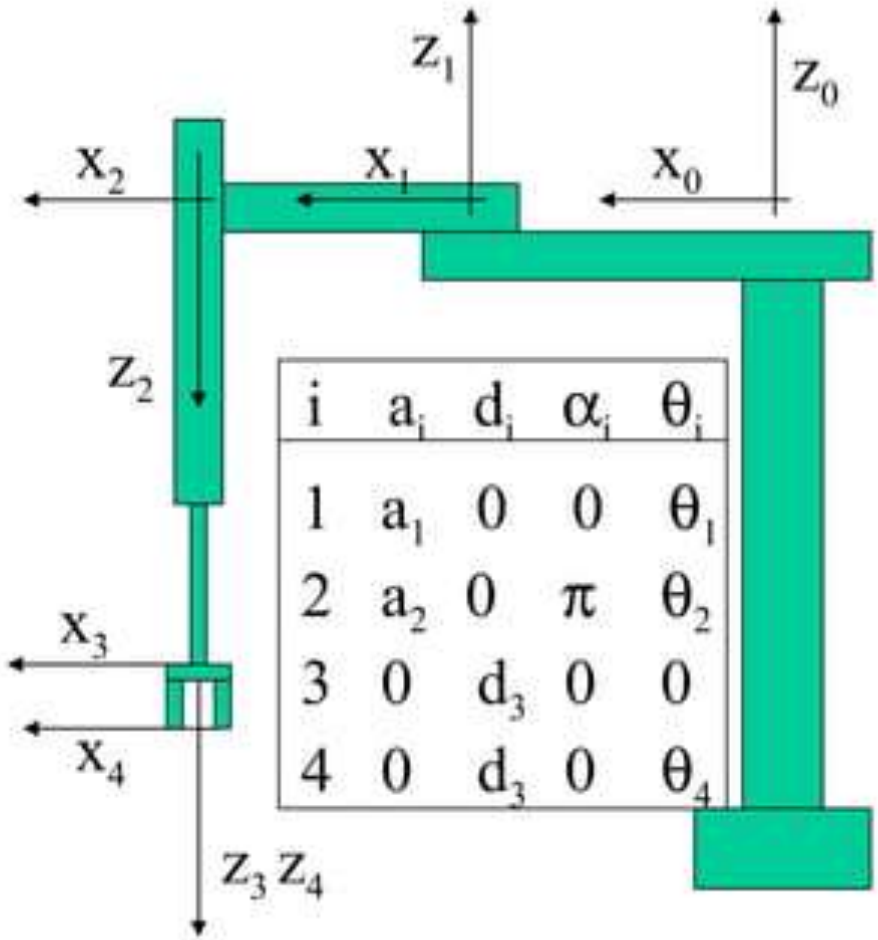
Examples

- 2- Planar Manipulator
  - Axes at distal ends
  - Avoid transformation to find endpoint
  - Make x_2 the approach vector



SCARA

- Note for joint 3 displacement is the joint variable and θ is constant
- Location of z_2 is arbtrary
- Location of O_3 and x_3 are arbitrary, however O_3 determines d_3
- For simplicity frame 4 is placed at the gripper and z_2 , z_3 and z_4 coincident



i	a_i	d_i	α_i	θ_i
1	a_1	0	0	θ_1
2	a_2	0	π	θ_2
3	0	d_3	0	0
4	0	d_3	0	θ_4

Forward Kinematics

- Link Coordinate Transform
  - From link i to link i-1of the 2-d planar robot
- Rotate about the z_{i-1} axis by θ_i , then about the x_i axis by α_i.

$$^{i-1}R_i = R_z(\theta)R_x(\alpha_i) = \begin{bmatrix} c\theta_i & -s\theta_i & 0 \\ s\theta_i & c\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha_i & -s\alpha_i \\ 0 & s\alpha_i & c\alpha_i \end{bmatrix} = \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i \\ 0 & s\alpha_i & c\alpha_i \end{bmatrix}$$

Forward Kinematics

- Translational component referenced to axes i-1

$$^{i-1}\mathbf{d}_{i-1,i} = d_i\,^{i-1}\mathbf{z}_{i-1} + a_i\,^{i-1}\mathbf{x}_i = d_i\,^{i-1}\mathbf{z}_{i-1} + a_i\,^{i-1}R_i\,^{i}\mathbf{x}_i = d_i\begin{bmatrix}0\\0\\1\end{bmatrix} + a_i\begin{bmatrix}c\theta_i\\s\theta_i\\0\end{bmatrix} = \begin{bmatrix}a_i c\theta_i\\a_i s\theta_i\\d_i\end{bmatrix}$$

- Homogeneous transform $^{i-1}T_i$ from frame i to frame i-1

$$^{i-1}T_i = \begin{bmatrix}^{i-1}R_i & ^{i-1}d_{i-1,i}\\0^T & 1\end{bmatrix} = \begin{bmatrix}c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i\\s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i\\0 & s\alpha_i & c\alpha_i & d_i\\0 & 0 & 0 & 1\end{bmatrix}$$

Relating any two link frames

- By composition of intervening frames we can link any two frames

$$^{i}T_{j} = {}^{i}T_{i+1}\,{}^{i+1}T_{i+2} \cdots {}^{j-2}T_{j-1}\,{}^{j-1}T_{j} \quad \mathrm{i} < \mathrm{j}$$

- For efficiency we normally decompose the frames into separate Rotations and Translation components

Forward Kinematic computations

- Aim: find the position and orientation of the last frame, n, wrt the base frame 0
- Sometimes additional frames are added at the beginning or end
  - Vision systems
  - Tools or object which are picked up
- Once an object is grasped the its kinematics are constant and for practical purposes can be considered part of the last link.

Kinematic Calibration

- Set during Manufacture
- Wear, error, etc…
- Kinematic callibration schemes
  - Measurement
  - Experimental
- Tool Transform
  - Most objects geometric and known
  - Vision system

Inverse Kinmatics (IK)

"Given a goal position find the joint angles for the robot arm"

Inverse Kinematics

- IK generally harder than FK
- Sometimes no analytical solution
- Sometimes multiple solutions
  - Redundant manipulators
- Sometimes no solution
  - Outside workspace

- 2-D planar manipulator (again)
- Solve for θ_2

$$x^2 + y^2 = a_1^2 + a_2^2 - 2a_1a_2\cos(\pi - \theta_2)$$

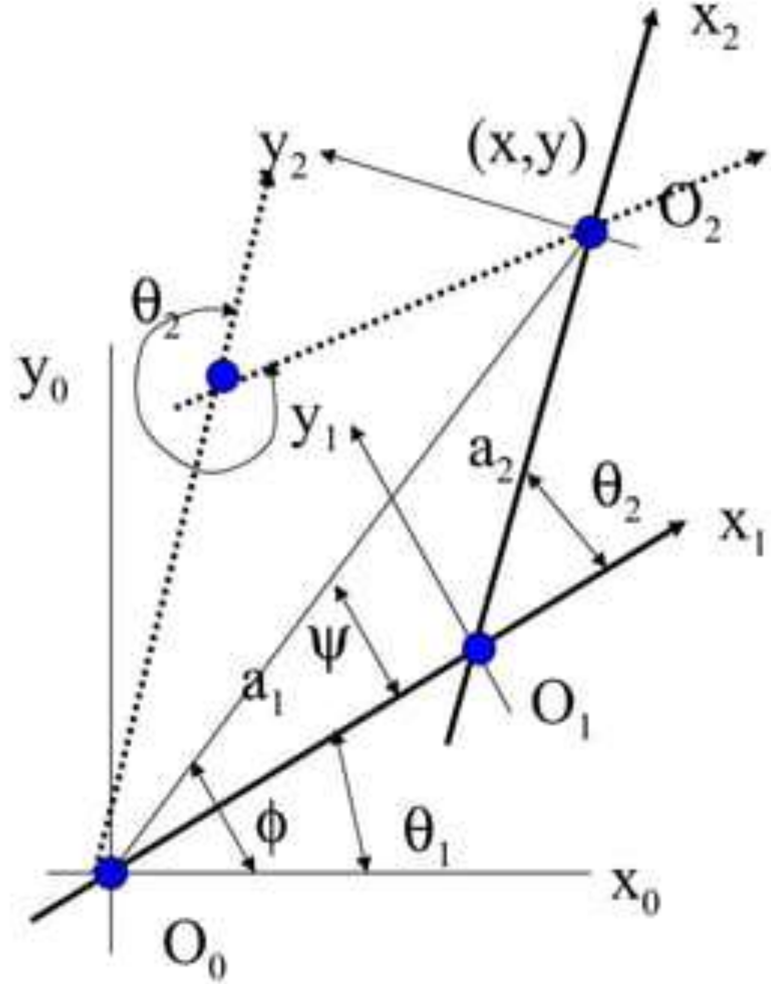
$$\cos\theta_2 = \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1a_2}$$

for greater accuracy

$$\tan^2\frac{\theta_2}{2} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{2a_1a_2 - x^2 - y^2 + a_1^2 + a_2^2}{2a_1a_2 + x^2 + y^2 - a_1^2 - a_2^2}$$

$$= \frac{\left(a_1^2 + a_2^2\right)^2 - \left(x^2 + y^2\right)}{\left(x^2 + y^2\right) - \left(a_1^2 - a_2^2\right)^2}$$

$$\theta_2 = \pm 2\tan^{-1}\sqrt{\frac{\left(a_1^2 + a_2^2\right)^2 - \left(x^2 + y^2\right)}{\left(x^2 + y^2\right) - \left(a_1^2 - a_2^2\right)^2}}$$

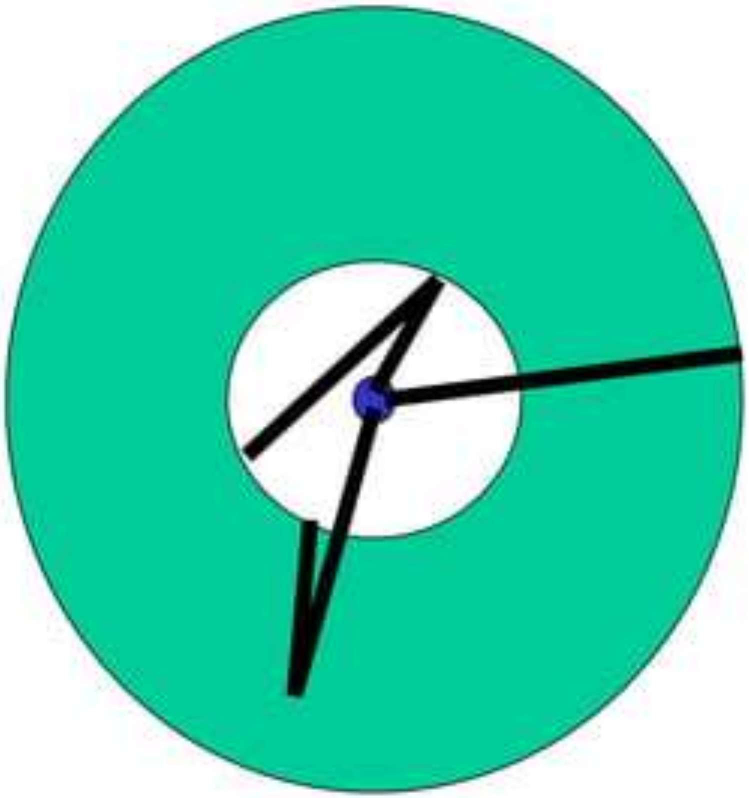


- Given θ_2 find θ_1

$$\theta_1 = \phi - \psi$$
$$\phi = a\tan 2(y, x)$$
$$\psi = a\tan 2(a_2 \sin\theta_2, a_1 + a_2 \cos\theta_2)$$

- Note that there are two answers for θ_1 based on elbow up or down
- 3-DOF of freedom robot arms
  - Most robots are made of two interconnected 3-DOF arms
    - Elbow joint (position wrist in space)
    - Wrist joint (orient object/tool in space)

Workspaces



Workspace limitations depend on
- Joint limits
- Presence of obstacles

Holes: doughnut shaped WS
Voids: empty space in WS

Total or Reachable Workspace
Primary WS: points reachable in all Orientations
Secondary WS: total -primary

Trajectory Planning

Path Planning

Trajectory planning

- Start to Goal avoiding obstacles along the way
- Joint space easiest because no IK
  - But end effector pose is not controlled
- Cartesian space planning is easier but IK must be solved

Joint Space Trajectories

- Cubic Trajectories
  - 4 coefficients
  - Satisfy position and velocity constraits
  - For a joint variable q_i

$$q_i(t_0) = q_0 \qquad q_i(t_f) = q_1$$

$$\dot{q}_i(t_0) = q_0' \qquad \dot{q}_i(t_f) = q_1'$$

t_0 is the start time $\qquad$ t_f is the end time

q_0 and q_0' are the initial position and velocity

q_1 and q_1' are the final position and velocity

- Polynomials for joint position and velocity

$$q_i(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$
$$\dot{q}_i(t) = a_1 + 2a_2 t + 3a_3 t^2$$

- The a_i coeff. have to be related to the end point constraits

$$t_0 = 0, \;\; q_i(0) = a_0 = q_0, \;\; \dot{q}_i(0) = a_1 = q_0'$$
$$q_i(t_f) = a_0 + a_1 t_f + a_2 t_f^{\;2} + a_3 t_f^{\;3} = q_1$$
$$\dot{q}_i(t_f) = a_1 + 2a_2 t_f + 3a_3 t_f^2 = q_1'$$

- This allows us to determine a_2 and a_3

$$a_2 = \frac{3(q_1 - q_0) - (2q_0' + q_1')t_f}{t_f^2}$$

$$a_3 = \frac{2(q_0 - q_1) + (q_0' + q_1')t_f}{t_f^3}$$

- Example

$$q_0 = 10^o, q_1 = -20^o, q_0' = q_1' = 0 \text{ deg/sec}, t_f = 1$$

$$\Rightarrow a_0 = 10, a_1 = 0, a_2 = -90, a_3 = 60$$

Linear segments with parabolic bends

- We want the middle part of the trajectory to have a constant velocity V
  - Ramp up
  - Linear segment
  - Ramp down

Ramp Up

- A quadratic requires 3 constraits
  - 2 for the start and 1 for const velocity at the end

$$q_i(t) = a_0 + a_1 t + a_2 t^2$$
$$\dot{q}_i(t) = a_1 + 2a_2 t$$
$$\text{with t} = 0 \text{ and velocity} = 0$$
$$a_0 = q_0, a_1 = q_0, a_2 \text{ found at next stage}$$
$$q_i(t) = q_0 + a_2 t^2$$
$$\dot{q}_i(t) = 2a_2 t$$

Linear Section

- Given constant velocity V, which we ramp up to in a unknown time t_b, we can find a_2 in terms of t_b

From end of ramp up : $\dot{q}_i(t) = 2a_2 t_b = V,\ a_2 = \dfrac{V}{2t_b}$

Linear segment : $q_i(t) = \alpha_0 + \alpha_1 t = \alpha_0 + Vt \quad t_b \le t \le t_f - t_b$

by symmetery : $q_i\left(t_f/2\right) = \dfrac{q_0 + q_1}{2} = \alpha_0 + \dfrac{Vt_f}{2}$

at time t_b $\quad q_0 + \dfrac{V}{2t_b} t_b^2 = \dfrac{q_0 + q_1 - Vt_f}{2} + Vt_b$

$$t_b = \frac{q_0 - q_1 + Vt_f}{V}$$

$$\text{Let } a = \ddot{q}_i = \frac{V}{t_b} \quad \Rightarrow \quad q_i(t) = \begin{cases} q_0 + \frac{1}{2}at^2 & 0 \le t \le t_b \\ \frac{q_0 + q_1 + Vt_f}{2} + Vt & t_b \le t \le t_f - t_b \end{cases}$$

- Similarly the end position can be show to be

$$q_i(t) = q_1 - \frac{a}{2}t_f^2 + at_f t - \frac{a}{2}t^2$$

$$\text{where}: q_i(t_f) = q_1 \text{ and } \dot{q}_i(t_f) = 0$$