Conditional probability:

The conditional probability of an event B assuming that the event A has happened, is defined as,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad \text{provided } P(A) \neq 0$$

Similarly,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad \text{provided } P(B) \neq 0$$

Theorem:

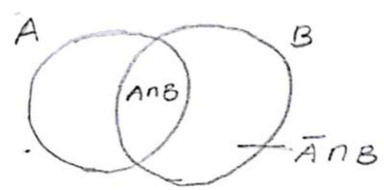
If A and B are independent, then prove that

1. $\bar{A}$ and B are independent
2. A and $\bar{B}$ are independent
3. $\bar{A}$ and $\bar{B}$ are independent.

Proof: Given A and B are independent $\Rightarrow P(A \cap B) = P(A)P(B) \longrightarrow (1)$

1. $P(\bar{A} \cap B) = P(\bar{A}) \cdot P(B)$

Consider,



$$B = (A \cap B) \cup (\bar{A} \cap B)$$

$$P(B) = P[(A \cap B) \cup (\bar{A} \cap B)]$$

$$P(B) = P(A \cap B) + P(\bar{A} \cap B)$$

$$\Rightarrow P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

$$= P(B) - P(A)P(B) \quad (\text{from } (1))$$

$$= P(B)[1 - P(A)]$$

$$\boxed{P(\bar{A} \cap B) = P(B) \cdot P(\bar{A})}$$

$\therefore \bar{A}$ and B are independent.

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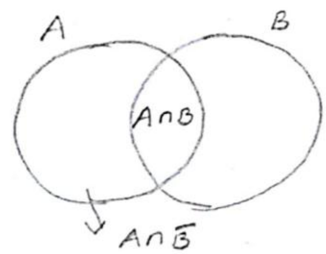
2. $\underline{P(A \cap \bar{B}) = P(A) \cdot P(\bar{B})}$



Consider,

$$A = (A \cap \bar{B}) \cup (A \cap B)$$

$$P(A) = P[(A \cap \bar{B}) \cup (A \cap B)]$$

$$= P(A \cap \bar{B}) + P(A \cap B)$$

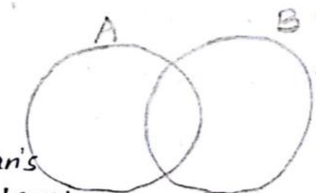
$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

$$= P(A) - P(A)P(B)$$

$$= P(A)[1 - P(B)]$$

$$\boxed{P(A \cap \bar{B}) = P(A)\, P(\bar{B})}$$ $\therefore$ A and $\bar{B}$ are independent

3. $\underline{P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B})}$



Consider,

$$P(\bar{A} \cap \bar{B}) = \overline{P(A \cup B)}$$ (by Demorgan's law)

$$= 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - [P(A) + P(B) - P(A) \cdot P(B)]$$

$$= 1 - P(A) - P(B) + P(A) \cdot P(B)$$

$$= P(\bar{A}) - P(B)[1 - P(A)]$$

$$= P(\bar{A}) - P(B) \cdot P(\bar{A})$$

$$= P(\bar{A})[1 - P(B)]$$

$$\boxed{P(\bar{A} \cap \bar{B}) = P(\bar{A})\, P(\bar{B})}$$

$\therefore$ $\bar{A}$ and $\bar{B}$ are independent.

PROBLEMS :

(1) From a bag containing 5 white balls and 6 green ba[lls]
3 balls are drawn with replacement. What is the chanc[e]
that (i) all are of same colour (ii) they are alternativel[y]
of different colours.

Solution:

$$S = \{5W, 6G\}$$

(i) P(all are of same colour)

$= P(\text{all are white or all are green})$

$= P(\text{all are white}) + P(\text{all are green})$

$= P(\text{I}W\ \text{II}W\ \text{III}W) + P(\text{I}G\ \text{II}G\ \text{III}G)$

$= \frac{5}{11} \times \frac{5}{11} \times \frac{5}{11} + \frac{6}{11} \times \frac{6}{11} \times \frac{6}{11}$

$= \frac{125}{1331} + \frac{216}{1331}$

$= \frac{341}{1331}$

(ii) P (they are alternatively of different colours)

$= P(\text{I}W\ \text{II}G\ \text{III}W \text{ or } \text{I}G\ \text{II}W\ \text{III}G)$

$= \frac{5}{11} \times \frac{6}{11} \times \frac{5}{11} + \frac{6}{11} \times \frac{5}{11} \times \frac{6}{11}$

$= \frac{150}{1331} + \frac{180}{1331}$

$= \frac{330}{1331}$

(2) If A and B are events with $P(A) = \frac{3}{8}$, $P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{4}$, find $P(A^c \cap B^c)$.

Soln:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{3}{8} + \frac{1}{2} - \frac{1}{4} = \frac{5}{8}$$

$$P(A^c \cap B^c) = P[(A \cup B)^c]$$
$$= 1 - P(A \cup B)$$
$$= 1 - \frac{5}{8} = \frac{3}{8}.$$

(3) If $P(A) = 0.4$, $P(B) = 0.7$, $P(A \cap B) = 0.3$, find $P(\bar{A} \cap \bar{B})$ & $P(\bar{A} \cup \bar{B})$.

Soln:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 0.4 + 0.7 - 0.3$$
$$= 0.8$$

$$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$$
$$= 1 - P(A \cup B)$$
$$= 1 - 0.8$$
$$= 0.2$$

$$P(\bar{A} \cup \bar{B}) = P(\overline{A \cap B})$$
$$= 1 - P(A \cap B)$$
$$= 1 - 0.3$$
$$= 0.7$$