NEWTON'S METHOD (or) NEWTON'S RAPHSON METHOD

Formula: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, provided $f'(x_n) \neq 0$

① Find the smallest positive root of the eqn. $x^3 - 2x + 0.5 = 0$.

Let $f(x) = x^3 - 2x + 0.5$; $f'(x) = 3x^2 - 2$

Now $f(0) = 0.5$ (+ve)

$f(1) = -0.5$ (−ve)

∴ The root lies btwn. 0 & 1.

Since $|f(0)| = |f(1)|$, let us assume $x_0 = 0$

Newton's Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

putting $n = 0$, $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$$= 0 - \frac{0.5}{-2} = 0.25$$

putting $n = 1$, $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 0.25 - \dfrac{f(0.25)}{f'(0.25)} = 0.2586$

putting $n = 2$, $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} = 0.2586 - \dfrac{f(0.2586)}{f'(0.2586)} = 0.2586$

Since x_2 & x_3 are equal root, the smallest positive root is 0.2586

② Compute the real root of $x \log_{10} x = 1.2$ correct to three decimal places using Newton's - Raphson Method.

Let $f(x) = x \log_{10} x - 1.2$; $f'(x) = \log_{10} x + 1$

$f(x) = 0.4343\, x \log_e x - 1.2$

$f'(x) = 0.4343 \log_e x + 0.4343$

$f(0) = -1.2$ (-ve)

$f(1) = -1.2$ (-ve)

$f(2) = -0.5980$ (-ve)

$f(3) = 0.2313$ (+ve)

∴ The root lies between 2 & 3.

Since $|f(2)| > |f(3)|$, let us assume $x_0 = 3$.

[since 0.2313 is nearer to '0' than 0.5980, we assume the root '3' as x_0.]

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.8436$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.7812$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.7567$$

$x_4 = 2.7469$

$x_5 = 2.7431$

$x_6 = 2.7416$

$x_7 = 2.7410$

$x_8 = 2.7407$

$x_9 = 2.7406$

$x_{10} = 2.7406$ Therefore the required root is 2.7406.

(3) Find the positive root of $2x^3 - 3x - 6 = 0$

Let $f(x) = 2x^3 - 3x - 6$; $f'(x) = 6x^2 - 3$

$f(0) = -6$ (-ve)

$f(1) = -7$ (-ve)

$f(2) = 4$ (+ve)

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$x_1 = 1.80952$

$x_2 = 1.78419$

$x_3 = 1.78377$

$x_4 = 1.78377$

Since x_3 & x_4 are equal, Therefore the required root is 1.78377

(4) Find the -ve root of $x^3 - \sin x + 1 = 0$. Radian mode

Let $f(x) = x^3 - \sin x + 1$; $f'(x) = 3x^2 - \cos x$. $\sin(-x) = -\sin x$

$f(-x) = -x^3 + \sin x + 1$

$f(-0) = 1$ (+ve)

$f(-1) = -1 + \sin 1 + 1 = 0.8414$ (+ve)

$f(-2) = -8 + \sin 8 + 1 = -6.0106$ (-ve)

$\therefore$ The root lies btwn -1 & -2.

Since $|f(-1)| < |f(-2)|$, let us assume $x_0 = -1$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -1.3421$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -1.2564$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = -1.2491$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = -1.2490$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = -1.2490$$

Since x_4 & x_5 are equal, therefore the required root is -1.2490.

① Obtain Newton's iterative formula for finding $\sqrt{N}$ where N is a +ve real no. Hence evaluate $\sqrt{5}$.

Soln. Let $x = \sqrt{N}$

$$x^2 = N$$

$$\Rightarrow x^2 - N = 0$$

$$f(x) = x^2 - N \; ; \; f'(x) = 2x$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \left(\frac{x_n^2 - N}{2x_n}\right)$$

$$= \frac{2x_n^2 - x_n^2 + N}{2x_n}$$

$= \dfrac{x_n^2 + N}{2x_n}$, is an iterative formula for $\sqrt{N}$.

To find $\sqrt{5}$.

$x = \sqrt{5}$

$x^2 - 5 = 0$

$\Rightarrow f(x) = x^2 - 5 \quad ; \quad f'(x) = 2x$

$f(0) = -5 \quad (-ve)$

$f(1) = -4 \quad (-ve)$

$f(2) = -1 \quad (-ve)$

$f(3) = 4 \quad (+ve)$

$\therefore$ The root lies btwn. 2 & 3.

Since $|f(2)| < |f(3)|$, let us assume $x_0 = 2$, (since the value 1 is nearer to '0' than 4)

$$\left\{ \begin{aligned} x_1 &= x_n - \frac{f(x_n)}{f'(x_0)} \\ &= 2 - \frac{(-1)}{4} = \frac{9}{4} = 2.25 \end{aligned} \right]$$

Here $x_{n+1} = \dfrac{x_n^2 + N}{2x_n}$.

$x_1 = \dfrac{x_0^2 + N}{2x_0}$

Here $N = 5$, $x_0 = 2$.

$$\Rightarrow x_1 = \frac{2^2+5}{2(2)} = \frac{9}{4} = 2.25$$

$$x_2 = \frac{x_1^2+N}{2x_1}$$
$$= \frac{(2.25)^2+5}{2(2.25)} = 2.2361$$

$$x_3 = \frac{x_2^2+N}{2x_2}$$
$$= \frac{(2.2361)^2+5}{2(2.2361)} = 2.2360$$

$$x_4 = 2.2360.$$

Since x_3 & x_4 are equal, the required root is 2.2360.

① Find the iterative formula for finding the value of $\frac{1}{N}$ (reciprocal of N) where N is a real no., using NRM. Hence evaluate $\frac{1}{26}$ correct to 4 decimal places.

Soln: Let $x = \frac{1}{N}$

(i.e) $N = \frac{1}{x}$.

Let $f(x) = \frac{1}{x} - N$; $f'(x) = -\frac{1}{x^2}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \left(\frac{\frac{1}{x_n} - N}{-\frac{1}{x_n^2}}\right)$$

$$= x_n + x_n^{\cancel{2}}\left(\frac{1-Nx_n}{\cancel{x_n}}\right)$$

$= x_n + x_n - N x_n^2$

$= 2x_n - N x_n^2$, is the Iterative formula.

To find $\frac{1}{26}$; $N = 26$.

$f(x) = \frac{1}{x} - 26$; $f'(x) = -\frac{1}{x^2}$

$f(0) = -26$ (-ve)

$f(1) = -25$ (-ve)

$f(2) = -25.5$ (-ve) [It's impossible to find the roots]

Let us take $x_0 = \frac{1}{25} = 0.04$, nearer to the given N

$x_0 = 0.04$.

WKT, $x_{n+1} = 2x_n - N x_n^2$.

$x_1 = 2x_0 - 26 x_0^2$

$= 2(0.04) - 26(0.04)^2$

$= 0.0384$

$x_2 = 0.0384$.

Since x_1 & x_2 are equal, the value of $\frac{1}{26} = 0.0384$

(2) Derive Newton's algorithm for finding the p^{th} root of a number N. & find the value of $(24)^{1/3}$

Soln: Let $x = N^{1/p}$.

$x^P = N$

$\Rightarrow x^P - N = 0$

Let $f(x) = x^P - N$; $f'(x) = px^{P-1}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \left(\frac{x_n^P - N}{Px_n^{P-1}} \right)$$

$$= \frac{Px_n^P - x_n^P + N}{Px_n^{P-1}} = \frac{(P-1)x_n^P + N}{Px_n^{P-1}}$$

To find $(24)^{1/3}$:

Here $N = 24$, $P = 3$.

$f(x) = x^P - N$

$f(x) = x^3 - 24$

$f(0) = -24$ (-ve)

$f(1) = -23$ (-ve)

$f(2) = -16$ (-ve)

$f(3) = 3$ (+ve), The root lies btwn. 2 & 3.

Since $|f(2)| > |f(3)|$, let us assume $x_0 = 3$.

$$x_{n+1} = \frac{(3-1)x_n^3 + 24}{3x_n^{3-1}} = \frac{2x_n^3 + 24}{3x_n^2}$$

$$x_1 = \frac{2x_0^3 + 24}{3x_0^2} = 2.8888.$$

$$x_2 = 2.8845$$

$$x_3 = 2.8844$$

$$x_4 = 2.8844.$$

Since $x_3 = x_4$, the required root is 2.8844.