SNS College of Technology(Autonomous)
Coimbatore-35
Academic Year 2023 – 2024 (Even)

UNIT 1 QUANTITATIVE ABILITY III

T1: Algebra, Indices and Surds

Indices and Surds

The term indices refers to the power to which a number is raised.

Thus x 2 is a number with an index of 2.

People prefer the phrase "x to the power of 2".

Term surds is not often used, instead term roots is used.

We know that 64 = 2 x 2 x 2 x 2 x 2 x 2 = 2 6 Here 2 is called the base and 6 is called the power (or index or exponent). We say that "64 is equal to base 2 raised to the power 6"

x^3 is a shortening of x. x. x . In the same way, any number to the power of n is that number multiplied by itself n times. To describe this in more detail, in the expression x^3, the x is referred to as the base, and the 3 as the exponent.

We know that 2 cubed is 2 × 2 × 2, and we say that we have 2 raised to the power 3, or to the index 3.

An easy way of writing this repeated multiplication is by using a ‘superscript’, so that we would write 2^3 : $2^3 = 2 \times 2 \times 2 = 8$.

Similarly, 4 cubed is 4 × 4 × 4, and equals 64. So we write $4^3 = 4 \times 4 \times 4 = 64$. But what if we have negative powers? What would be the value of 4 −3?

To find out, we shall look at what we know already: $4^3 = 4 \times 4 \times 4 = 64$, $4^2 = 4 \times 4 = 16$, 4 1 = 4 = 4 , and so 4 0 = 4 ÷ 4 = 1 (because to get the answer you divide the previous one by 4).

Now let’s continue the pattern: 4 −1 = 1 ÷ 4 = 1 4 , 4 −2 = 1 4 ÷ 4 = 1 16 , 4 −3 = 1 16 ÷ 4 = 1 64

The term "surd" is used to name any number which involves <u>non exact</u> square roots.

Surds are Irrational Numbers

Simple surds: $\sqrt{2}$ $\sqrt{29}$ $\sqrt[3]{10}$

Other surds: $3\sqrt{2}$ $5\sqrt{3}$ $2\sqrt[3]{7}$ $\frac{\sqrt{10}}{\sqrt{3}}$

$\sqrt{2} + 1$ $\sqrt{5} - \sqrt{2}$ $\frac{2\sqrt{10} - 9}{1 + \sqrt{3}}$

The Rules of Roots and Surds

$$\sqrt{xy} = \sqrt{x}\sqrt{y}$$

e.g. $\sqrt{12} = \sqrt{3}\sqrt{4}$, $\sqrt{6}\sqrt{5} = \sqrt{30}$

$$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$$

e.g. $\sqrt{\frac{10}{3}} = \frac{\sqrt{10}}{\sqrt{3}}$, $\sqrt{\frac{50}{21}} = \frac{\sqrt{50}}{\sqrt{21}}$

adding/subtracting:

$$a\sqrt{x} \pm b\sqrt{x} = (a \pm b)\sqrt{x}$$

e.g. $3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2}$

$8\sqrt{7} - 2\sqrt{7} = 6\sqrt{7}$

Note

Although we are showing these rules with square roots, they work with all roots, i.e. cube roots, fourth roots etc

The Rules of Roots and Surds

$$\sqrt{xy} = \sqrt{x}\sqrt{y}$$

$$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$$

$$\sqrt{xy} = (xy)^{\frac{1}{2}} = x^{\frac{1}{2}}\, y^{\frac{1}{2}} = \sqrt{x}\sqrt{y}$$

$$\sqrt{\frac{x}{y}} = \left[\frac{x}{y}\right]^{\frac{1}{2}} = \frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}} = \frac{\sqrt{x}}{\sqrt{y}}$$

adding/subtracting:

$$a\sqrt{x} \pm b\sqrt{x} = (a \pm b)\sqrt{x}$$

$$c(a \pm b) = ca \pm cb$$

Why do these rules work?

The Rules of Roots and Surds

$$\sqrt{xy} = \sqrt{x}\sqrt{y}$$

$$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$$

WARNING !

$$\sqrt{x} + \sqrt{y} \neq \sqrt{x+y}$$

$$\sqrt{4} + \sqrt{9} \neq \sqrt{13} \longleftarrow \approx 3.6$$

$$2 + 3 = 5$$

adding/subtracting:

$$a\sqrt{x} \pm b\sqrt{x} = (a \pm b)\sqrt{x}$$

Surd operations

$$\left(2+\sqrt{3}\right)\left(\sqrt{2}-1\right)=\sqrt{6}-\sqrt{3}+2\sqrt{2}-2$$

evaluate: $\left(2+\sqrt{2}\right)\left(3-\sqrt{2}\right)$

$$\left(2+\sqrt{2}\right)\left(3-\sqrt{2}\right) = 6 - 2\sqrt{2} + 3\sqrt{2} - 2$$

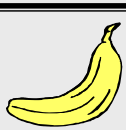
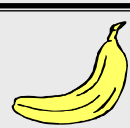
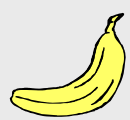
$$\sqrt{2} \times \sqrt{2} = 2$$

$$\sqrt{3} \times \sqrt{3} = 3$$

$$\sqrt{7.77} \times \sqrt{7.77} = 7.77$$

$$\sqrt{\frac{4}{5}} \times \sqrt{\frac{4}{5}} = \frac{4}{5}$$

$$\sqrt{p} \times \sqrt{p} = p$$

$\sqrt{}$  $\times$ $\sqrt{}$  $=$ 

evaluate: $(2+\sqrt{2})(3-\sqrt{2})$

$$(2+\sqrt{2})(3-\sqrt{2}) = 6 - 2\sqrt{2} + 3\sqrt{2} - 2 = 4+\sqrt{2}$$

we treat surds like algebraic quantities

Expand the following brackets and simplify your answer as much as possible:

$$(1+\sqrt{5})(3-\sqrt{5}) = 3-\sqrt{5}+3\sqrt{5}-5 = 2\sqrt{5}-2$$

$$(1+2\sqrt{2})(\sqrt{2}-1) = \sqrt{2}-1+4-2\sqrt{2} = 3-\sqrt{2}$$

$$(4+\sqrt{2})(2-\sqrt{3}) = 8-4\sqrt{3}+2\sqrt{2}-\sqrt{6} = 8-\sqrt{6}-4\sqrt{3}+2\sqrt{2}$$

$$(\sqrt{3}-\sqrt{2})(2\sqrt{2}-\sqrt{3}) = 2\sqrt{6}-3-4+\sqrt{6} = 3\sqrt{6}-7$$

$$(5+2\sqrt{7})(2\sqrt{7}-5) = \cancel{10\sqrt{7}}-25+28-\cancel{10\sqrt{7}} = 3$$

Common manipulations involving surds are to write them in terms of smaller surds.
This usually involves application of the three basic rules:

$$\sqrt{xy} = \sqrt{x}\sqrt{y}$$

$$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$$

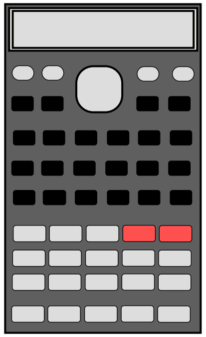
$$a\sqrt{x} \pm b\sqrt{x} = (a \pm b)\sqrt{x}$$

Simplify $\sqrt{8}$

$$\sqrt{8} = \sqrt{4' \; 2} = \sqrt{4}' \sqrt{2} = 2' \sqrt{2} = 2\sqrt{2}$$

Split 8 into two factors
One must be an exact root

Check!



Common manipulations involving surds are to write them in terms of smaller surds.
This usually involves application of the three basic rules:

$$\sqrt{xy} = \sqrt{x}\sqrt{y}$$

$$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$$

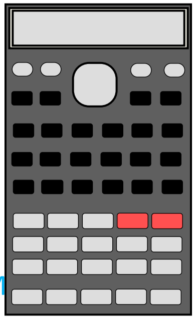
$$a\sqrt{x} \pm b\sqrt{x} = (a \pm b)\sqrt{x}$$

Simplify $\sqrt{8}$ $\quad \sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2 \times \sqrt{2} = 2\sqrt{2}$

Simplify $\sqrt{20}$ $\quad \sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2 \times \sqrt{5} = 2\sqrt{5}$

Split 20 into two factors
One must be an exact root

Check!



**Common manipulations involving surds are to write them in terms of smaller surds.
This usually involves application of the three basic rules:**

$$\sqrt{xy} = \sqrt{x}\sqrt{y}$$

$$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$$

$$a\sqrt{x} \pm b\sqrt{x} = (a \pm b)\sqrt{x}$$

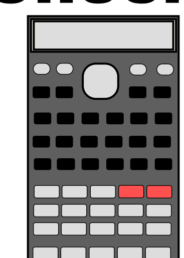
Simplify $\sqrt{8}$ $\quad \sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2 \times \sqrt{2} = 2\sqrt{2}$

Simplify $\sqrt{20}$ $\quad \sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2 \times \sqrt{5} = 2\sqrt{5}$

Simplify $\sqrt{45}$ $\quad \sqrt{45} = \sqrt{9 \times 5} = \sqrt{9} \times \sqrt{5} = 3 \times \sqrt{5} = 3\sqrt{5}$

Split 45 into two factors
One must be an exact root

Check!



Common manipulations involving surds are to write them in terms of smaller surds.
This usually involves application of the three basic rules:

$$\sqrt{xy} = \sqrt{x}\sqrt{y}$$

$$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$$

$$a\sqrt{x} \pm b\sqrt{x} = (a \pm b)\sqrt{x}$$

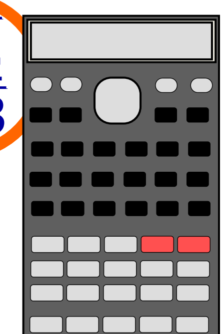
Simplify $\sqrt{8}$ $\quad \sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2 \times \sqrt{2} = 2\sqrt{2}$

Simplify $\sqrt{20}$ $\quad \sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2 \times \sqrt{5} = 2\sqrt{5}$

Simplify $\sqrt{45}$ $\quad \sqrt{45} = \sqrt{9 \times 5} = \sqrt{9} \times \sqrt{5} = 3 \times \sqrt{5} = 3\sqrt{5}$

Simplify $\sqrt{\frac{98}{75}}$ $\quad \sqrt{\frac{98}{75}} = \frac{\sqrt{98}}{\sqrt{75}} = \frac{\sqrt{49 \times 2}}{\sqrt{25 \times 3}} = \frac{\sqrt{49} \times \sqrt{2}}{\sqrt{25} \times \sqrt{3}} = \frac{7\sqrt{2}}{5\sqrt{3}}$

Check!



Common manipulations involving surds are to write them in terms of smaller surds.
This usually involves application of the three basic rules:

$$\sqrt{xy} = \sqrt{x}\sqrt{y}$$

$$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$$

$$a\sqrt{x} \pm b\sqrt{x} = (a \pm b)\sqrt{x}$$

Simplify $\sqrt{8} + \sqrt{32} = \sqrt{4 \times 2} + \sqrt{16 \times 2}$

$= \sqrt{4} \times \sqrt{2} + \sqrt{16} \times \sqrt{2}$

$= 2\sqrt{2} + 4\sqrt{2}$

$= 6\sqrt{2}$

Common manipulations involving surds are to write them in terms of smaller surds. This usually involves application of the three basic rules:

$$\sqrt{xy} = \sqrt{x}\sqrt{y}$$

$$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$$

$$a\sqrt{x} \pm b\sqrt{x} = (a \pm b)\sqrt{x}$$

Simplify $\sqrt{45} - \sqrt{20} = \sqrt{9 \times 5} - \sqrt{4 \times 5}$

$= \sqrt{9} \times \sqrt{5} - \sqrt{4} \times \sqrt{5}$

$= 3\sqrt{5} - 2\sqrt{5}$

$= \sqrt{5}$

Common manipulations involving surds are to write them in terms of smaller surds.
This usually involves application of the three basic rules:

$$\sqrt{xy} = \sqrt{x}\sqrt{y}$$

$$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$$

$$a\sqrt{x} \pm b\sqrt{x} = (a \pm b)\sqrt{x}$$

Simplify $\sqrt{48} - \sqrt{75} + \sqrt{50} - \sqrt{12}$

$$= \sqrt{16 \times 3} - \sqrt{25 \times 3} + \sqrt{25 \times 2} - \sqrt{4 \times 3}$$

$$= \sqrt{16} \times \sqrt{3} - \sqrt{25} \times \sqrt{3} + \sqrt{25} \times \sqrt{2} - \sqrt{4} \times \sqrt{3}$$

$$= 4\sqrt{3} - 5\sqrt{3} + 5\sqrt{2} - 2\sqrt{3}$$

$$= -3\sqrt{3} + 5\sqrt{2}$$

$$= 5\sqrt{2} - 3\sqrt{3}$$

rationalising surd denominators

$$\frac{\sqrt{50}-1}{\sqrt{8}+3} = 17\sqrt{2} - 23$$

In higher level mathematics, we usually want the denominators of fractions surd-free.

Some of the reasons for this are:

- **Easier to add fractions with rational denominators**
- **Easier to get a "feel" for the size of a fraction with a surd-free denominator**
- **Looks!**

Given a fraction with a surd appearing in the denominator, we can find another equivalent fraction with the denominator surd-free.

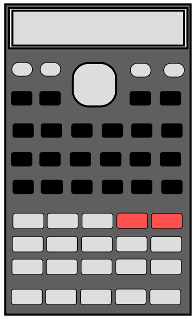
We are <u>rationalising the denominator</u>

Here are some of the "tricks" involved in this process:

Rationalising Denominators with simple Surds

Rationalise the denominator of $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

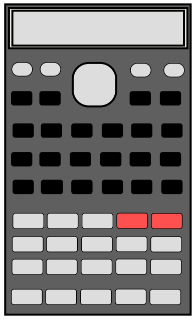
Check!



$$\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{\sqrt{2}}{2}$$

Rationalise the denominator of $\frac{3}{\sqrt{3}} = \sqrt{3}$

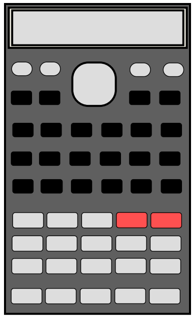
Check!



$$\frac{3}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{3 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{\cancel{3}\sqrt{3}}{\cancel{3}} = \sqrt{3}$$

Rationalise the denominator of $\frac{3}{2\sqrt{5}} = \frac{3}{10}\sqrt{5}$

Check!

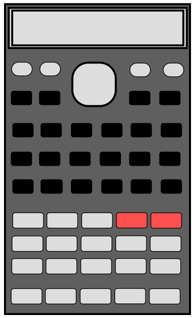


$$\frac{3}{2\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{3 \times \sqrt{5}}{2\sqrt{5} \times \sqrt{5}} = \frac{3\sqrt{5}}{10} = \frac{3}{10}\sqrt{5}$$

Rationalising Denominators with not so simple Surds

Rationalise the denominator of $\frac{1}{3+\sqrt{2}} = \frac{3-\sqrt{2}}{7}$

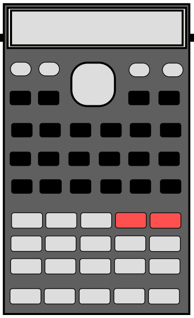
Check!



$$\frac{1}{3+\sqrt{2}} \times \frac{3-\sqrt{2}}{3-\sqrt{2}} = \frac{3-\sqrt{2}}{9-3\sqrt{2}+3\sqrt{2}-2} = \frac{3-\sqrt{2}}{7}$$

$$(a+b)\times(a-b) = a^2 - b^2$$

Check!



Rationalise the denominator of $\frac{3}{\sqrt{5}-2} = 3\sqrt{5}+6$

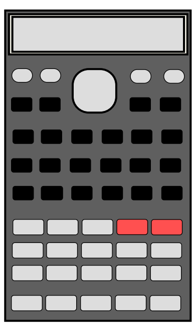
$$\frac{3}{\sqrt{5}-2} \times \frac{\sqrt{5}+2}{\sqrt{5}+2} = \frac{3\times(\sqrt{5}+2)}{5+2\sqrt{5}-2\sqrt{5}-4} = \frac{3\sqrt{5}+6}{1} = 3\sqrt{5}+6$$

Rationalising Denominators with not so simple Surds

Rationalise the denominator of $\dfrac{\sqrt{2}}{\sqrt{3}+\sqrt{2}} = \sqrt{6}-2$

$$\frac{\sqrt{2}}{\sqrt{3}+\sqrt{2}} \times \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}} = \frac{\sqrt{2}\times\left(\sqrt{3}-\sqrt{2}\right)}{3-\sqrt{6}+\sqrt{6}-2} = \frac{\sqrt{6}-2}{1} = \sqrt{6}-2$$

$$(a+b)\times(a-b) = a^2-b^2$$



Check!

Rationalising Denominators with not so simple Surds

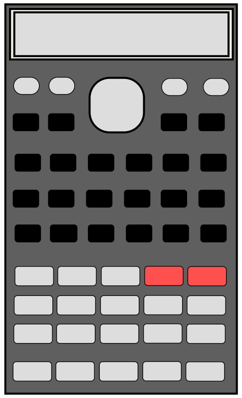
An A*/A-Level example

Rationalise the denominator of $\dfrac{\sqrt{2}}{3\sqrt{2}-2\sqrt{3}} = \dfrac{\sqrt{6}+3}{3}$

$$\frac{\sqrt{2}}{3\sqrt{2}-2\sqrt{3}} \times \frac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}} = \frac{\sqrt{2}\times\left(3\sqrt{2}+2\sqrt{3}\right)}{\left(3\sqrt{2}\right)^2-\left(2\sqrt{3}\right)^2} = \frac{6+2\sqrt{6}}{18-12} = \frac{\not{2}\left(3+\sqrt{6}\right)}{\not{6}_3}$$

$$(a-b) \times (a+b) \qquad = a^2 - b^2$$

Check!



$$= \frac{\sqrt{6}+3}{3}$$

$$= \tfrac{1}{3}\left(\sqrt{6}+3\right)$$

$$= \frac{\sqrt{6}}{3} + 1$$

How?