Wave parameters

Plane waves in Lossless medium:

In a lossless medium, ε and μ are real numbers, so k is real.

In Cartesian coordinates each of the equations 6.1(a) and 6.1(b) are equivalent to three scalar Helmholtz's equations, one each in the components E_x, E_y and E_z or H_x , H_y, H_z.

For example if we consider E_x component we can write

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0 \qquad \text{..(6.2)}$$

A uniform plane wave is a particular solution of Maxwell's equation assuming electric field (and magnetic field) has same magnitude and phase in infinite planes perpendicular to the direction of propagation. It may be noted that in the strict sense a uniform plane wave doesn't exist in practice as creation of such waves are possible with sources of infinite extent. However, at large distances from the source, the wavefront or the surface of the constant phase becomes almost spherical and a small portion of this large sphere can be considered to plane. The characteristics of plane waves are simple and useful for studying many practical scenarios.

Let us consider a plane wave which has only E_x component and propagating along z .
Since the plane wave will have no variation along the plane perpendicular to z i.e., xy plane, $\frac{\partial E_x}{\partial x} = \frac{\partial E_x}{\partial y} = 0$. The Helmholtz's equation (6.2) reduces to,

$$\frac{d^2 E_x}{dz^2} + k^2 E_x = 0 \qquad \text{..(6.3)}$$

The solution to this equation can be written as

$$E_x(z) = E_x^+(z) + E_x^-(z)$$
$$= E_0^+ e^{-jkz} + E_0^- e^{jkz}$$

$E_0^+ \;\&\; E_0^-$ are the amplitude constants (can be determined from boundary conditions).In the time domain,

$$\varepsilon_X(z,t) = \mathrm{Re}(E_x(z)e^{jwt})$$

$$\varepsilon_X(z,t) = E_0^+ \cos(\omega t - kz) + E_0^- \cos(\omega t + kz)$$

as. $E_0^+ \;\&\; E_0^-$ are real constants.

Here $\varepsilon_X^+(z,t) = E_0^+ \cos(\omega t - \beta z)$ represents the forward traveling wave. Theplot of $\varepsilon_X^+(z,t)$
for several values of t is shown in the Figure.)

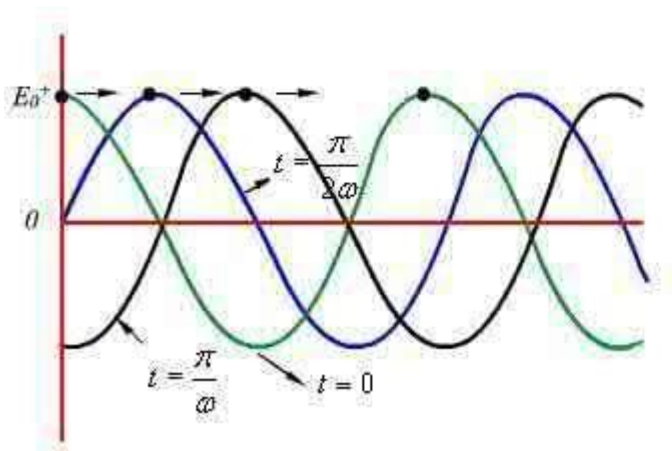


Figure: Plane wave traveling in the + *z* direction

As can be seen from the figure, at successive times, the wave travels in the +z direction.

If we fix our attention on a particular point or phase on the wave (as shownby the dot) i.e. , $\omega t - kz$ = constant

Then we see that as *t* is increased to $t + \Delta t$, z also should increase to $z + \Delta z$ so that

$$\omega(t + \Delta t) - k(z + \Delta z) = \text{constant} = \omega t - \beta z$$

Or, $\omega \Delta t = k \Delta z$

Or, $\frac{\Delta z}{\Delta t} = \frac{\omega}{k}$ When $\Delta t \to 0$,

we write $\lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \frac{dz}{dt}$ = phase velocity v_p . $\therefore v_p = \frac{\omega}{k}$ ....(6.6)

$\varepsilon = \varepsilon_0$, $\mu = \mu_0$

If the medium in which the wave is propagating is free space i.e.,

Then $$v_p = \frac{\omega}{\omega\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = C$$

Where '*C*' is the speed of light. That is plane EM wave travels in free space with thespeed of light.

5.1. Intrinsic impedance

The wavelength λ is defined as the distance between two successive maxima (or minimaor any other reference points).

i.e., $(\omega t - kz) - [\omega t - k(z + \lambda)] = 2\pi$

or, $k\lambda = 2\pi$

or, $\lambda = \frac{2\pi}{k}$

Substituting $k = \frac{\omega}{v_p}$,

$$\lambda = \frac{2\pi v_p}{2\pi f} = \frac{v_p}{f}$$

or, ...$\lambda f = v_p$................................ (6.7)

Thus wavelength λ also represents the distance covered in one oscillation

of the wave. Similarly, $\varepsilon^-(z,t) = E_0^- \cos(\omega t + kz)$ represents a plane wave traveling inthe -z direction.

The associated magnetic field can be found as follows:

From (6.4),

$$\vec{E}_x^+(z) = E_0^+ e^{-jkz}\hat{a}_x$$

$$\vec{H} = -\frac{1}{j\omega\mu}\nabla\times\vec{E}$$

$$= -\frac{1}{j\omega\mu}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_0^+ e^{-jkz} & 0 & 0 \end{vmatrix}$$

$$= \frac{k}{\omega\mu}E_0^+ e^{-jkz}\hat{a}_y$$

$$\frac{E_0^+}{\eta}e^{-jkz}\hat{a}_y = H_0^+ e^{-jkz}\hat{a}_y$$

=........... (6.8)

where $\eta = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}$ is the intrinsic impedance of the medium.When the wave travels in free space

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cong 120\pi = 377\Omega$$

is the intrinsic impedance of the free space.

In the time domain,

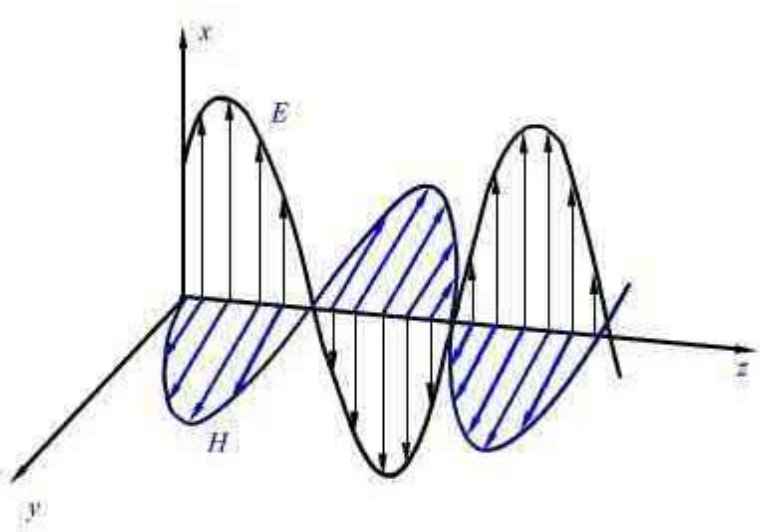
$$\vec{H}^+(z,t) = \hat{a}_y\frac{E_0^+}{\eta}\cos(\omega t - \beta z)$$

Which represents the magnetic field of the wave traveling in the +z direction.For the negative traveling wave,

$$\overrightarrow{H}^{-}(z,t) = -a_y \frac{E_0^{+}}{\eta} \cos(\omega t + \beta z)$$ ....(6.10)

For the plane waves described, both the E & H fields are perpendicular to the direction ofpropagation, and these waves are called TEM (transverse electromagnetic) waves.

The *E* & *H* field components of a TEM wave is shown in Fig 6.2.



E & H fields of a particular plane wave at time t.

So far we have considered a plane electromagnetic wave propagating in the z-direction. Let us now consider the propagation of a uniform plane wave in any arbitrary direction that doesn't necessarily coincides with an axis.

For a uniform plane wave propagating in z-direction

$\vec{E}(z) = E_0 e^{-jkz}$, E_0 is a constant vector........................... (6.11)

The more general form of the above equation is

$\vec{E}(x,y,z) = \vec{E}_0 e^{-jk_x x - jk_y y - jk_z z}$ .... (6.12)

This equation satisfies Helmholtz's equation

$\nabla^2 \vec{E} + k^2 \vec{E} = 0$ provided,

$k_x^2 + k_y^2 + k_z^2 = k^2 = \omega^2 \mu \varepsilon$ .... (6.13)

We define wave number vector $\vec{k} = \hat{a}_x k_x + \hat{a}_y k_y + \hat{a}_z k_z = k \hat{a}_n$ ....(6.14)

And radius vector from the origin

$\vec{r} = \hat{a}_x x + \hat{a}_y y + \hat{a}_z z$ .... (6.15)

Therefore we can write

$\vec{E}(\vec{r}) = \vec{E}_0 e^{-j\vec{k}\cdot\vec{r}} = \vec{E}_0 e^{-jk\hat{a}_n\cdot\vec{r}}$)

Here $\hat{a}_n.\vec{r}$ =constant is a plane of constant phase and uniform amplitude just in the case of $\vec{E}(z) = \vec{E}_0 e^{-jkz}$,

z =constant denotes a plane of constant phase and uniform amplitude.If the region under consideration is charge free,

$$\nabla.\vec{E} = 0$$

$$\therefore \nabla.\left(\vec{E}_0 e^{-j\vec{k}.\vec{r}}\right) = 0$$

Using the vector identity $\nabla.\left(f\vec{A}\right) = \vec{A}\nabla f + f\nabla.\vec{A}$ and noting that $\vec{E}_0$ is constant we canwrite,

$$\vec{E}_0.\nabla\left(e^{-jk\hat{a}_n.\vec{r}}\right) = 0$$

$$or,\ \vec{E}_0.\left[\left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)e^{-j(k_x x + k_y y + k_z z)}\right] = 0$$

$$or,\ \vec{E}_0.\left(-jk\hat{a}_n e^{-jk\hat{a}_n.\vec{r}}\right) = 0$$

$\vec{E}_0.\hat{a}_n = 0$ ....(6.17)

i.e., $\vec{E}_0$ is transverse to the direction of the propagation.

The corresponding magnetic field can be computed as follows:

$$\vec{H}\left(\vec{r}\right) = -\frac{1}{j\omega\mu}\nabla\times\vec{E}\left(\vec{r}\right) = -\frac{1}{j\omega\mu}\nabla\times\left(\vec{E}_0 e^{-j\vec{k}.\vec{r}}\right)$$

Using the vector identity,

$$\nabla\times\left(\psi\vec{A}\right)=\psi\nabla\times\vec{A}+\nabla\psi\times\vec{A}$$

Since $\vec{E}_0$ is constant we can write,

$$\vec{H}(\vec{r})=-\frac{1}{j\omega\mu}\nabla e^{-j\vec{k}.\vec{r}}\times\vec{E}_0$$
$$=-\frac{1}{j\omega\mu}\left[-jk\hat{a}_n\times\vec{E}_0e^{-jk\hat{a}_n.\vec{r}}\right]$$
$$=\frac{k}{\omega\mu}\hat{a}_n\times\vec{E}\left(\vec{r}\right)$$

$$\vec{H}(\vec{r})=\frac{1}{\eta}\hat{a}_n\times\vec{E}\left(\vec{r}\right)$$ (6.18)

Where η is the intrinsic impedance of the medium. We observe that $\vec{H}(\vec{r})$ isperpendicular to both $\hat{a}_n$ and $\vec{E}\left(\vec{r}\right)$. Thus the electromagnetic wave represented by $\vec{E}\left(\vec{r}\right)$ and is a TEM wave.

5.1.1 propagation constant

In a lossy medium, the EM wave looses power as it propagates. Such a medium is conducting with conductivity σ and we can write:

$$\nabla\times\vec{H}=\vec{J}+j\omega\varepsilon\vec{E}=\left(\sigma+j\omega\varepsilon\right)\vec{E}$$
$$=j\omega\left(\varepsilon+\frac{\sigma}{j\omega}\right)\vec{E}$$
$$=j\omega\varepsilon_c\vec{E}$$ ....(6.19)

Where $\varepsilon_c=\varepsilon-j\frac{\sigma}{\omega}=\varepsilon'-j\varepsilon''$ is called the complex permittivity.

We have already discussed how an external electric field can polarize a dielectric and give rise to bound charges. When the external electric field is time varying, the polarization vector will vary with the same frequency as that of the applied field. As the frequency of the applied filed increases, the inertia of the charge particles tend to prevent the particle displacement keeping pace with the applied field changes. This results in frictional damping mechanism causing power loss.

In addition, if the material has an appreciable amount of free charges, there will be ohmiclosses. It is customary to include the effect of damping and ohmic losses in the imaginary part of ε . An equivalent conductivity $\sigma = \omega \varepsilon''$ represents all losses.

The ratio $\frac{\varepsilon''}{\varepsilon'}$ is called loss tangent as this quantity is a measure of the power loss.

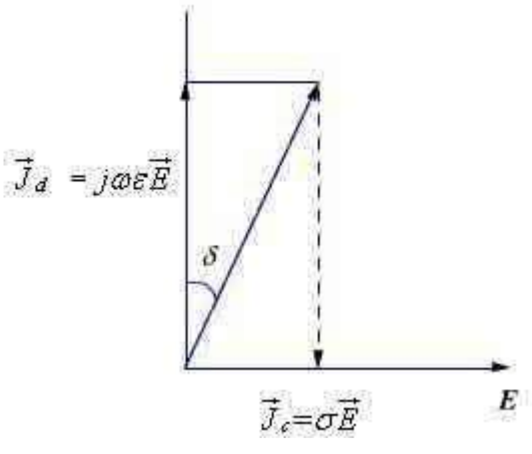


Fig 6.3 : Calculation of Loss Tangent

With reference to the Fig 6.3,

$$\tan\delta = \frac{|\vec{J}_c|}{|\vec{J}_d|} = \frac{\sigma}{\omega\varepsilon} = \frac{\varepsilon''}{\varepsilon'} \qquad \text{....................... (6.20)}$$

where $\vec{J}_c$ is the conduction current density and $\vec{J}_d$ is displacement current density. Theloss tangent gives a measure of how much lossy is the medium under consideration. For a good dielectric medium $(\sigma << \omega\varepsilon)$, $\tan\delta$ is very small and the medium is a good conductor if $(\sigma \gg \omega\varepsilon)$. A material may be a goodconductor at low frequencies but behave as lossy dielectric at higher frequencies.

For a source free lossy medium we can write

$$\left.\begin{array}{ll}\nabla\times\vec{H} = (\sigma + j\omega\varepsilon)\vec{E} & \nabla.\vec{H} = 0 \\ \nabla\times\vec{E} = -j\omega\mu\vec{H} & \nabla.\vec{E} = 0\end{array}\right\} \quad \text{........................ (6.21)}$$

$$\nabla\times\nabla\times\vec{E} = \nabla\left(\nabla.\vec{E}\right) - \nabla^2\vec{E} = -j\omega\mu\nabla\times\vec{H} = -j\omega\mu(\sigma + j\omega\varepsilon)\vec{E}$$

$$or, \nabla^2\vec{E} - \gamma^2\vec{E} = 0$$

.... (6.22)

Where $\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$

Proceeding in the same manner we can write,

$$\nabla^2\vec{H} - \gamma^2\vec{H} = 0$$

$$\gamma = \alpha + \imath\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = j\omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)^{1/2}$$

is called the propagation constant.

The real and imaginary parts α and β of the propagation constant γ can be computed asfollows:

$$\gamma^2 = (\alpha + j\beta)^2 = j\omega\mu(\sigma + j\omega\varepsilon)$$

$$or,\ \alpha^2 - \beta^2 = -\omega^2\mu\varepsilon$$

And $\alpha\beta = \frac{\omega\mu\sigma}{2}$

$$\therefore \alpha^2 - \left(\frac{\omega\mu\sigma}{2\alpha}\right)^2 = -\omega^2\mu\varepsilon$$

$$or,\ 4\alpha^4 + 4\alpha^2\omega^2\mu\varepsilon = \omega^2\mu^2\sigma^2$$

$$or,\ 4\alpha^4 + 4\alpha^2\omega^2\mu\varepsilon + \omega^4\mu^2\varepsilon^2 = \omega^2\mu^2\sigma^2 + \omega^4\mu^2\varepsilon^2$$

$$or,\ \left(2\alpha^2 + \omega^2\mu\varepsilon\right)^2 = \omega^4\mu^2\varepsilon^2\left(1 + \frac{\sigma^2}{\omega^2\varepsilon^2}\right)$$

$$or,\ \alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]}$$ .... (6.23a)

Similarly.................. $$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]}$$ (6.23b)

Let us now consider a plane wave that has only x -component of electric field andpropagate along z .

$$\therefore \vec{E}_x(z) = \left(E_0^+ e^{-\gamma z} + E_0^- e^{\gamma z}\right)\hat{a}_x$$ .... (6.24)

Considering only the forward traveling wave

$$\vec{\mathcal{E}}(z,t) = \text{Re}\left(E_0^+ e^{-\gamma z} e^{j\omega t}\right)\hat{a}_x$$

$$= E_0^+ e^{-\alpha z}\cos(\omega t - \beta z)\hat{a}_x$$ .... (6.25)

Similarly, from $\vec{\mathcal{H}} = -\frac{1}{j\omega\mu}\nabla\times\vec{E}$, we can find

$$\vec{\mathcal{H}}(z,t) = \frac{E_0}{\eta}e^{-\alpha z}\cos(\omega t - \beta z)\hat{a}_y$$ .... (6.26)

Where $\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta| e^{j\theta_\eta}$

$$\therefore \vec{\mathcal{H}} = \frac{E_0}{|\eta|}e^{-\alpha z}\cos(\omega t - \beta z - \theta_\eta)\hat{a}_y$$ .... (6.27)

From (6.25) and (6.26) we find that as the wave propagates along z, it decreases in amplitude by a factor $e^{-\alpha z}$. Therefore α is known as attenuation constant.Further $\vec{E}$ and $\vec{H}$ are out of phase by an angle θ_η.

For low loss dielectric, $\frac{\sigma}{\omega\varepsilon} << 1$, i.e., $\varepsilon'' << \varepsilon'$.

Using the above condition approximate expression for α and β can be obtained asfollows:

$$\gamma = \alpha + i\beta = j\omega\sqrt{\mu\varepsilon'}\left[1 - j\frac{\varepsilon''}{\varepsilon'}\right]^{1/2}$$

$$\cong j\omega\sqrt{\mu\varepsilon'}\left[1 - j\frac{1}{2}\frac{\varepsilon''}{\varepsilon'} + \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right]$$

$$\left.\begin{aligned}\alpha &= \frac{\omega\varepsilon''}{2}\sqrt{\frac{\mu}{\varepsilon'}}\\ \beta &= \omega\sqrt{\mu\varepsilon'}\left[1+\frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right]\end{aligned}\right\}\text{.......}(6.28)$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon'}}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2}$$

$$= \sqrt{\frac{\mu}{\varepsilon'}}\left(1 + j\frac{\varepsilon''}{2\varepsilon'}\right)$$

` .9)

& phase velocity

$$v_p = \frac{\omega}{\beta} \cong \frac{1}{\sqrt{\mu\varepsilon'}}\left[1 - \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right]$$

.... (6.30)

For good conductors $\frac{\sigma}{\omega\varepsilon} >> 1$

$$\gamma = j\omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma}{j\omega\varepsilon}\right) \cong j\omega\sqrt{\mu\varepsilon}\sqrt{\frac{\sigma}{j\omega\varepsilon}}$$

$$\frac{1+j}{\sqrt{2}}\sqrt{\omega\mu\sigma}$$

=............... (6.31)

We have used the relation

$$\sqrt{j} = \left(e^{j\pi/2}\right)^{1/2} = e^{j\pi/4} = \frac{1}{\sqrt{2}}(1+j)$$

From (6.31) we can write

$$\alpha + i\beta = \sqrt{\pi f \mu \sigma} + j\sqrt{\pi f \mu \sigma}$$

$$\therefore \alpha = \beta = \sqrt{\pi f \mu \sigma}$$ .... (6.32)

$$\eta = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon\left(1+\frac{\sigma}{j\omega\varepsilon}\right)}}$$

$$\cong \sqrt{\frac{\mu}{\varepsilon}\frac{j\omega\varepsilon}{\sigma}} = \sqrt{\frac{j\omega\mu}{\sigma}}$$

$$= (1+j)\sqrt{\frac{\pi f \mu}{\sigma}}$$

$$= (1+j)\frac{\alpha}{\sigma}$$ .... (6.33)

And phase velocity

$$v_p = \frac{\omega}{\beta} \cong \sqrt{\frac{2\omega}{\mu\sigma}}$$ .... (6.34)