



UNIT 5 - Multiple Integrals

Applications of multiple integrals are to find areas and volume of various bodies just by taking a little part of them into consideration.

In probability theory, it is used to evaluate probabilities of two dimensional continuous random variables

Double Integration (Cartesian coordinate)

A double integral is computed by repeated single variable integration, integrate w.r.t one variable treating the other variable as constant.

$$I = \iint_R f(x, y) dx dy \text{ where } R \text{ is the Region.}$$

Formulae:

$$1. \int x^n dx = \frac{x^{n+1}}{n+1}$$

$$2. \int k dx = kx$$

$$3. \int \frac{1}{x} dx = \log x$$

$$4. \int e^x dx = e^x$$

$$5. \int \sin x dx = -\cos x$$

$$6. \int \cos x dx = \sin x$$

$$7. \int \tan x dx = \log(\sec x) \text{ (or)} \\ - \log(\cos x)$$

$$8. \int \cot x dx = \log(\sin x) = \log(\csc x)$$

$$9. \int \csc x dx = \log[\csc x - \cot x]$$

$$10. \int \sec x dx = \log[\sec x + \tan x]$$

$$11. \int \sec^2 x dx = \tan x$$

$$12. \int \csc^2 x dx = -\cot x.$$



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13) $\int \sec x \tan x \, dx = \sec x$ 20. $\int \frac{1}{1+x^2} \, dx = \tan^{-1} x$
 14. $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x$ 21. $\int \frac{-1}{1+x^2} \, dx = \tan^{-1} x$
 15. $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \left[\frac{x}{a} \right]$ 22. $\int \frac{1}{x\sqrt{x^2-1}} \, dx = \operatorname{sec}^{-1} x$
 16. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \left(\frac{x-a}{x+a} \right)$ 23. $\int \frac{-1}{x\sqrt{x^2-1}} \, dx = \operatorname{cosec}^{-1} x$
 17. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \left(\frac{x}{a} \right)$ 24. $\int \sqrt{a^2-x^2} \, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2 \sin^{-1} \frac{x}{a}}{2}$
 18. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$ 25. $\int \frac{dx}{\sqrt{a^2+x^2}} = \log [x + \sqrt{x^2+a^2}]$
 19. $\int \frac{-1}{\sqrt{1-x^2}} \, dx = \cos^{-1} x$

26. $\int u \, dv = uv - \int v \, du$ 27. $\int u^n \, dx = \frac{u^{n+1}}{n+1} + \dots$ (Bernoulli's formula)

Problems on double integration in cartesian coordinates

II 1. Evaluate $\int_0^1 \int_0^2 x(x+y) \, dy \, dx$

Soln:

Now $\int_0^1 \int_0^2 x(x+y) \, dy \, dx = \int_0^1 \left[(x^2 + xy) \, dy \right] dx$
 $= \int_0^1 \left[x^2 y + x \frac{y^2}{2} \right]_{y=0}^2 dx$
 $= \int_0^1 \left[(2x^2 + \frac{4x}{2}) - (x^2 + \frac{x}{2}) \right] dx$
 $= \int_0^1 [2x^2 + 2x - x^2 - \frac{x}{2}] dx$
 $= \int_0^1 [x^2 + \frac{3}{2}x] dx$
 $= \left[\frac{x^3}{3} + \frac{3}{2} \frac{x^2}{2} \right]_{x=0}^1 = \left[\left(\frac{1}{3} + \frac{3}{4} \right) - 0 \right]$

$\therefore \int_0^1 \int_0^2 x(x+y) \, dy \, dx = \frac{13}{12}$



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2. Evaluate $\int_0^a \int_0^b xy(x-y) dx dy$

Soln:-
$$\int_0^a \int_0^b xy(x-y) dx dy = \int_0^a \int_0^b [x^2y - xy^2] dx dy$$
$$= \int_0^a \left[\frac{x^3y}{3} - \frac{x^2y^2}{2} \right]_{x=0}^b dy$$
$$= \int_0^a \left[\frac{b^3y}{3} - \frac{b^2y^2}{2} \right] dy$$
$$= \left[\frac{b^3}{3} \left(\frac{y^2}{2} \right) - \frac{b^2}{2} \left(\frac{y^3}{3} \right) \right]_{y=0}^a$$
$$= \left[\frac{b^3a^2}{6} - \frac{b^2a^3}{6} \right] - 0$$

$$\int_0^a \int_0^b xy(x-y) dx dy = \frac{a^2b^2}{6} [b-a]$$

3. Evaluate $\int_2^3 \int_1^2 \frac{1}{xy} dx dy$

Soln:-
Now
$$\int_2^3 \int_1^2 \frac{1}{xy} dx dy = \int_2^3 \int_1^2 \frac{1}{x} \cdot \frac{1}{y} dx dy$$
$$= \int_2^3 \frac{1}{y} [\log x]_{x=1}^2 dy$$
$$= \int_2^3 \frac{1}{y} [\log 2 - \log 1] dy$$
$$= \int_2^3 \frac{1}{y} \log 2 dy \quad [\because \log 1 = 0]$$
$$= \log 2 \int_2^3 \frac{dy}{y} = \log 2 [\log y]_2^3$$
$$\int_2^3 \int_1^2 \frac{1}{xy} dx dy = \log 2 \log \left(\frac{3}{2} \right) = \log 2 [\log 3 - \log 2] = \log 2 \log \left(\frac{3}{2} \right)$$



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4. Evaluate $\int_0^3 \int_0^2 e^{x+y} dy dx$.

$$\begin{aligned}\int_0^3 \int_0^2 e^{x+y} dy dx &= \int_0^3 \int_0^2 e^x \cdot e^y dy dx = \int_0^3 e^x [e^y]_{y=0}^2 dx \\ &= \int_0^3 e^x [e^2 - e^0] dx = [e^2 - 1] [e^x]_{x=0}^3 \\ &= [e^2 - 1] [e^3 - e^0] = [e^2 - 1] [e^3 - 1]\end{aligned}$$

$\therefore \int_0^3 \int_0^2 e^{x+y} dy dx = [e^3 - 1] [e^2 - 1]$

5. Evaluate $\int_0^5 \int_0^{x^2} x(x^2 + y^2) dx dy$

Soln: $\int_0^5 \int_0^{x^2} x(x^2 + y^2) dx dy = \int_0^5 \int_0^{x^2} [x^3 + xy^2] dy dx$

$$\begin{aligned}&= \int_0^5 [x^3 y + x \frac{y^3}{3}]_{y=0}^{x^2} dx \\ &= \int_0^5 [x^5 + \frac{x^7}{3}] - (0+0) dx \\ &= \int_0^5 [x^5 + \frac{x^7}{3}] dx \\ &= [\frac{x^6}{6} + \frac{x^8}{24}]_0^5 \\ &= (\frac{5^6}{6} + \frac{5^8}{24}) - 0 \\ &= 5^6 [\frac{29}{24}]\end{aligned}$$

$\int_0^5 \int_0^{x^2} x(x^2 + y^2) dx dy = 5^6 [\frac{29}{24}]$



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6. Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} dy dx$

Soln:- $\int_0^a \int_0^{\sqrt{a^2-x^2}} dy dx = \int_0^a [y]_{y=0}^{\sqrt{a^2-x^2}} dx$

$= \int_0^a \sqrt{a^2-x^2} dx$

$= \left[\frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) \right]_0^a$

$= (0 + \frac{a^2}{2} \sin^{-1}(1)) - (0+0)$

$= \frac{a^2}{2} \left(\frac{\pi}{2}\right) = \frac{\pi a^2}{4}$

7. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dxdy}{1+x^2+y^2}$

Soln:- $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dxdy}{1+x^2+y^2} = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy}{1+x^2+y^2} dx$

$= \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy}{y^2 + (\sqrt{1+x^2})^2} dx$

$= \int_0^1 \left(\frac{1}{\sqrt{1+x^2}} \tan^{-1} \left[\frac{y}{\sqrt{1+x^2}} \right] \right)_{y=0}^{\sqrt{1+x^2}} dx$

$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}(1) - 0 \right] dx$

$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left(\frac{\pi}{4}\right) dx = \frac{\pi}{4} \int_0^1 \frac{dx}{\sqrt{1+x^2}}$

$= \frac{\pi}{4} \left[\log(x + \sqrt{x^2+1^2}) \right]_0^1$



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$$= \frac{\pi}{4} [\log(1 + \sqrt{2}) - \log(0 + 1)]$$
$$= \frac{\pi}{4} \log(1 + \sqrt{2})$$

8. Evaluate $\int_0^a \int_0^{\sqrt{ay}} xy \, dx \, dy$

Soln:-

$$\int_0^a \int_0^{\sqrt{ay}} xy \, dx \, dy = \int_0^a \left[\frac{x^2}{2} y \right]_{x=0}^{\sqrt{ay}} dy$$
$$= \int_0^a \left[\frac{ay}{2} y - 0 \right] dy$$
$$= \int_0^a \frac{ay^2}{2} dy$$
$$= \frac{a}{2} \left[\frac{y^3}{3} \right]_0^a$$
$$= \frac{a}{2} \left[\frac{a^3}{3} \right]$$
$$= \frac{a^4}{6}$$
$$\int_0^a \int_0^{\sqrt{ay}} xy \, dx \, dy = \frac{a^4}{6}$$