



UNIT 5 - Multiple Integrals

Applications: Volume as triple integrals and solids of revolution
Problems based on volume.

1. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$ by triple integration

Given: $x^2 + y^2 + z^2 = a^2$

$$x^2 = a^2 - y^2 - z^2$$

$$x = \pm \sqrt{a^2 - y^2 - z^2}$$

Limits

$$x: 0 \text{ to } \sqrt{a^2 - y^2 - z^2}$$

$$y: 0 \text{ to } \sqrt{a^2 - z^2}$$

$$z: 0 \text{ to } a$$

Take $x=0$,

$$y^2 + z^2 = a^2$$

$$y^2 = a^2 - z^2$$

$$y = \pm \sqrt{a^2 - z^2}$$

Take $x=0, y=0$

$$z^2 = a^2$$

$$z = \pm a$$

Volume of the sphere = $8 \times$ Volume in the 1st octant

$$= 8 \iiint dx dy dz$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2 - z^2}} \int_0^{\sqrt{a^2 - y^2 - z^2}} dx dy dz$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2 - z^2}} [x]_0^{\sqrt{a^2 - y^2 - z^2}} dy dz$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2 - z^2}} [\sqrt{a^2 - y^2 - z^2} - 0] dy dz$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2 - z^2}} \sqrt{(a^2 - z^2) - y^2} dy dz$$



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$$\begin{aligned} &= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{(a^2-x^2)-y^2} \, dy \, dx \\ &= 8 \int_0^a \left[\frac{y\sqrt{a^2-x^2-y^2}}{2} + \frac{(a^2-x^2)}{2} \sin^{-1} \left(\frac{y}{\sqrt{a^2-x^2}} \right) \right]_0^{\sqrt{a^2-x^2}} dx \\ &= 8 \int_0^a \left[\left(0 + \frac{a^2-x^2}{2} \sin^{-1} \left(\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}} \right) \right) - (0+0) \right] dx \\ &= 8 \int_0^a \frac{a^2-x^2}{2} \left(\frac{\pi}{2} \right) dx \\ &= 2\pi \int_0^a (a^2-x^2) dx = 2\pi \left[a^2x - \frac{x^3}{3} \right]_0^a \\ &= 2\pi \left[\left(a^3 - \frac{a^3}{3} \right) - 0 \right] \\ &= 2\pi \left(\frac{2a^3}{3} \right) \\ &= \frac{4\pi a^3}{3} \\ \therefore \text{Volume of the sphere} &= \frac{4}{3} \pi a^3 \end{aligned}$$



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2. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Solution:-

$$\text{Given } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2} - \frac{z^2}{c^2}$$

$$x^2 = a^2 \left[1 - \frac{y^2}{b^2} - \frac{z^2}{c^2} \right]$$

$$x = \pm a \sqrt{1 - \frac{y^2}{b^2} - \frac{z^2}{c^2}}$$

Take $x=0$.

$$\frac{y^2}{b^2} = 1 - \frac{z^2}{c^2}$$

$$y^2 = b^2 \left(1 - \frac{z^2}{c^2} \right)$$

$$y = \pm b \sqrt{1 - \frac{z^2}{c^2}}$$

Take $x=0, y=0$.

$$\frac{z^2}{c^2} = 1$$

$$z^2 = c^2$$

$$z = \pm c$$

Limits:

$$x : 0 \text{ to } a \sqrt{1 - \frac{y^2}{b^2} - \frac{z^2}{c^2}}$$

$$y : 0 \text{ to } b \sqrt{1 - \frac{z^2}{c^2}}$$

$$z : 0 \text{ to } c$$



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$$\begin{aligned} \text{Volume} &= 8 \times \text{Volume in the 1st quadrant} \\ &= 8 \iiint dx dy dz \\ &= 8 \int_0^c \int_0^{b\sqrt{1-\frac{z^2}{c^2}}} \int_0^{a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}} dx dy dz \\ &= 8 \int_0^c \int_0^{b\sqrt{1-\frac{z^2}{c^2}}} [x]_0^{a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}} dy dz \\ &= 8 \int_0^c \int_0^{b\sqrt{1-\frac{z^2}{c^2}}} a \sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}} dy dz \\ &= 8a \int_0^c \int_0^{b\sqrt{1-\frac{z^2}{c^2}}} \sqrt{\left(1-\frac{z^2}{c^2}\right) - \frac{y^2}{b^2}} dy dz \\ &= 8a \int_0^c \int_0^{b\sqrt{1-\frac{z^2}{c^2}}} \sqrt{\frac{b^2\left(1-\frac{z^2}{c^2}\right) - y^2}{b^2}} dy dz \\ &= \frac{8a}{b} \int_0^c \int_0^{b\sqrt{1-\frac{z^2}{c^2}}} \sqrt{b^2\left(1-\frac{z^2}{c^2}\right) - y^2} dy dz \\ &= \frac{8a}{b} \int_0^c \left[\frac{y}{2} \sqrt{b^2\left(1-\frac{z^2}{c^2}\right) - y^2} + \frac{b^2\left(1-\frac{z^2}{c^2}\right)}{2} \sin^{-1} \frac{y}{b\sqrt{1-\frac{z^2}{c^2}}} \right]_0^{b\sqrt{1-\frac{z^2}{c^2}}} dz \\ &= \frac{8a}{b} \int_0^c \left[0 + \frac{b^2\left(1-\frac{z^2}{c^2}\right)}{2} \sin^{-1} \frac{b\sqrt{1-\frac{z^2}{c^2}}}{b\sqrt{1-\frac{z^2}{c^2}}} \right] dz \end{aligned}$$



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$$\begin{aligned} &= \frac{8a}{b} \int_0^c \frac{b^2}{2} \left(1 - \frac{z^2}{c^2}\right) \sin^{-1}(1) dz \\ &= 4ab \int_0^c \left(1 - \frac{z^2}{c^2}\right) \frac{\pi}{2} dz \\ &= 2\pi ab \int_0^c \left(1 - \frac{z^2}{c^2}\right) dz \\ &= 2\pi ab \left[z - \frac{z^3}{3c^2} \right]_0^c \\ &= 2\pi ab \left[c - \frac{c^3}{3c^2} \right] \\ &= 2\pi ab \left[c - \frac{c}{3} \right] \\ &= \frac{4}{3} \pi abc \end{aligned}$$

\therefore Volume of ellipsoid = $\frac{4}{3} \pi abc$.

3. Find the volume of the tetrahedron bound by the

plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Given: $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$$\frac{x}{a} = 1 - \frac{y}{b} - \frac{z}{c}$$

$$x = a \left(1 - \frac{y}{b} - \frac{z}{c} \right)$$

Take $x=0$,

$$\frac{y}{b} + \frac{z}{c} = 1$$

$$\frac{y}{b} = 1 - \frac{z}{c} \quad y = b \left(1 - \frac{z}{c} \right)$$





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Take $x=0, y=0$.

$$\frac{z}{c} = 1$$
$$z = c$$

Limits: $x : 0$ to $a(1 - \frac{y}{b} - \frac{z}{c})$

$$y : 0$$
 to $b(1 - \frac{z}{c})$
$$z : 0$$
 to c

Volume = $\iiint dx dy dz$

$$= \int_0^c \int_0^{b(1-\frac{z}{c})} \int_0^{a(1-\frac{y}{b}-\frac{z}{c})} dx dy dz$$
$$= \int_0^c \int_0^{b(1-\frac{z}{c})} [x]_0^{a(1-\frac{y}{b}-\frac{z}{c})} dy dz$$
$$= \int_0^c \int_0^{b(1-\frac{z}{c})} [a(1 - \frac{y}{b} - \frac{z}{c})] dy dz$$
$$= a \int_0^c \int_0^{b(1-\frac{z}{c})} [(1 - \frac{z}{c}) - \frac{y}{b}] dy dz$$
$$= a \int_0^c \left[(1 - \frac{z}{c})y - \frac{y^2}{2b} \right]_0^{b(1-\frac{z}{c})} dz$$
$$= a \int_0^c \left[(1 - \frac{z}{c})b(1 - \frac{z}{c}) - \frac{b^2(1 - \frac{z}{c})^2}{2b} \right] dz$$



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$$\begin{aligned} &= a \int_0^c \left[\left(1 - \frac{x}{c}\right)^2 b - \frac{b}{2} \left(1 - \frac{x}{c}\right)^2 \right] dx \\ &= a \int_0^c \frac{b}{2} \left(1 - \frac{x}{c}\right)^2 dx \\ &= \frac{ab}{2} \int_0^c \left(1 - \frac{x}{c}\right)^2 dx = \frac{ab}{2} \int_0^c \left(1 + \frac{x^2}{c^2} - \frac{2x}{c}\right) dx \\ &= \frac{ab}{2} \left[\frac{\left(1 - \frac{x}{c}\right)^3}{(-1/c)^3} \right]_0^c = \frac{ab}{2} \left[x + \frac{x^3}{3c^2} - \frac{2x^2}{c} \right]_0^c \\ &= \frac{-abc}{6} [0 - 1] = \frac{ab}{2} \left[c + \frac{c^3}{3c^2} - \frac{c^2}{c} - 0 \right] \\ &= \frac{abc}{6} = \frac{ab}{2} \left[c + \frac{c}{3} - c \right] \\ &= \frac{abc}{6} \end{aligned}$$

$\therefore \text{Volume} = \frac{abc}{6}$