



UNIT 5 - Multiple Integrals

Problems based on Area as a double Integral in Cartesian coordinates:

1. Find the area enclosed by the curves $y^2 = 4x$ and $x^2 = 4y$

Given that
 $y^2 = 4x \rightarrow \textcircled{1}$
 $x^2 = 4y$
 $y = \frac{x^2}{4} \rightarrow \textcircled{2}$

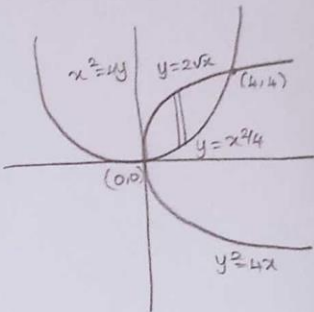
Sub y in $\textcircled{1}$
 $\left(\frac{x^2}{4}\right)^2 = 4x$
 $\frac{x^4}{16} = 4x$
 $x^3 = 64$
 $x = 4$

Subs $x = 4$ in $\textcircled{2}$
 $y = \frac{x^2}{4}$
 $= \frac{16}{4}$
 $y = 4$

Intersection points is $(4, 4)$

Limits:
 $x: 0$ to 4
 $y: \frac{x^2}{4}$ to $2\sqrt{x}$

We know that
 $A = \iint_R dy dx = \int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} dy dx$



$dy dx \rightarrow$ Vertical strip
 y limit \rightarrow intervals of x
 x limit \rightarrow Constant limits



UNIT 5 - Multiple Integrals

$$\begin{aligned}
 &= \int_0^4 [y]_{y=x^2/4}^{2\sqrt{x}} dx = \int_0^4 \left[2\sqrt{x} - \frac{x^2}{4} \right] dx \\
 &= \int_0^4 \left[2x^{1/2} - \frac{x^2}{4} \right] dx = \left[\frac{2 \cdot 2 x^{3/2}}{3/2} - \frac{x^3}{4 \times 3} \right]_0^4 \\
 &= \left[\frac{4}{3} x^{3/2} - \frac{x^3}{12} \right]_0^4 = \left[\frac{4}{3} (4)^{3/2} - \frac{(4)^3}{12} - 0 \right] \\
 &= \left[\frac{4}{3} \cdot (8) - \frac{64}{12} \right] = \frac{32}{3} - \frac{16}{3} = \frac{16}{3}.
 \end{aligned}$$

2. Find the smallest area bounded by
 $y = 2 - x$, $x^2 + y^2 = 4$

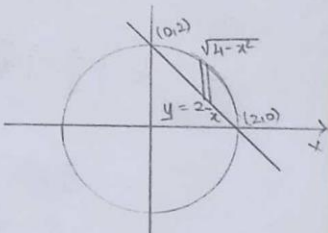
Solution:-
 Given that $y = 2 - x \rightarrow ①$
 $x^2 + y^2 = 4 \rightarrow ②$

Sub ① in ②
 $x^2 + (2-x)^2 = 4$
 $x^2 + 4 + x^2 - 4x = 4$
 $2x^2 - 4x = 0$
 $2x(x-2) = 0$
 $x = 0, x = 2$

When $x = 0$, $y = 2 - 0 = 2$
 $x = 2$, $y = 2 - 2 = 0$

\therefore The intersection points are $(0, 2)$ & $(2, 0)$

Limits : x : 0 to 2
 y : $2 - x$ to $\sqrt{4 - x^2}$

$$\text{Area} = \iint_R dy dx = \int_0^2 \int_{2-x}^{\sqrt{4-x^2}} dy dx$$


$dy dx \rightarrow$ Vertical strip
 y limit \rightarrow integrals of x
 x limit \rightarrow Constant limits
 $x^2 + y^2 = 4$
 $y^2 = 4 - x^2$
 $y = \pm \sqrt{4 - x^2}$



UNIT 5 - Multiple Integrals

$$\begin{aligned}
&= \int_0^2 [y]_{2-x}^{\sqrt{4-x^2}} dx \\
&= \int_0^2 [\sqrt{4-x^2} - (2-x)] dx \\
&= \int_0^2 \sqrt{4-x^2} dx - \int_0^2 (2-x) dx \\
&= \left[\frac{x}{2} \sqrt{4-x^2} + \frac{2^2}{2} \sin^{-1}\left(\frac{x}{2}\right) \right]_0^2 - \left[2x - \frac{x^2}{2} \right]_0^2 \\
&= \left[\frac{2}{2} \sqrt{4-2^2} + 2 \sin^{-1}\left(\frac{2}{2}\right) \right] - 0 - \left[4 - \frac{4}{2} \right] \\
&= 2 \sin^{-1}(1) - [4-2] \\
&= 2 \left(\frac{\pi}{2}\right) - 2 \quad \boxed{A = \pi - 2}
\end{aligned}$$

Q. Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

using double integration

Given that:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$= \frac{a^2 - x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2} (a^2 - x^2)$$

$$y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$$

$dy dx \rightarrow$ Vertical strip
 y limit \rightarrow in terms of x
 x limit \rightarrow constant limit

Limits
 $x : 0$ to a
 $y : 0$ to $\frac{b}{a} \sqrt{a^2 - x^2}$



UNIT 5 – Multiple Integrals

The area of the ellipse is

$$\begin{aligned} A &= 4 \times \text{Area in the first quadrant} \\ &= 4 \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dy dx \\ &= 4 \int_0^a \left[y \right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx \\ &= 4 \int_0^a \left[\frac{b}{a}\sqrt{a^2-x^2} - 0 \right] dx \\ &= \frac{4b}{a} \int_0^a \sqrt{a^2-x^2} dx \\ &= \frac{4b}{a} \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right]_0^a \\ &= \frac{4b}{a} \left[\frac{a}{2}\sqrt{a^2-a^2} + \frac{a^2}{2}\sin^{-1}\frac{a}{a} \right] \\ &= \frac{4b}{a} \left[\frac{a^2}{2} \left(\frac{\pi}{2} \right) \right] = \pi ab \end{aligned}$$

4. Find the area of a circle of radius a by double integration

Equation of the circle is

$$\begin{aligned} x^2 + y^2 &= a^2 \\ y^2 &= a^2 - x^2 \\ y &= \pm \sqrt{a^2 - x^2} \end{aligned}$$

$dy dx \rightarrow$ vertical strip
 y limit \rightarrow in terms of x
 x limit \rightarrow Constant limits



UNIT 5 – Multiple Integrals

Limits:

$$x : 0 \text{ to } a$$

$$y : 0 \text{ to } \sqrt{a^2 - x^2}$$

The area of the circle is

$A = 4 \times$ Area in the 1st quadrant

$$= 4 \int_0^a \int_0^{\sqrt{a^2 - x^2}} dy dx$$

$$= 4 \int_0^a [y]_0^{\sqrt{a^2 - x^2}} dx$$

$$= 4 \int_0^a [\sqrt{a^2 - x^2} - 0] dx$$

$$= 4 \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^a$$

$$= 4 \left[\left[0 + \frac{a^2}{2} \sin^{-1} 1 \right] - 0 \right]$$

$$= 2a^2 \left[\frac{\pi}{2} \right]$$

$$A = \pi a^2.$$