Triple Integration in cartesian coordinates:

Triple integration of a function defined over a region is

$$\iiint_R f(x, y, z)\, dx\, dy\, dz$$

Note:

$\iiint_R dx\,dy\,dz \to$ volume of the region R.

Problems:

1. Evaluate $\int_0^2 \int_1^3 \int_1^2 xy^2 z\, dz\, dy\, dx$

Soln.

$$\int_0^2 \int_1^3 \int_1^2 xy^2 z\, dz\, dy\, dx = \int_0^2 \int_1^3 x\, y^2 \left(\frac{z^2}{2}\right)_1^2 dy\, dx.$$

$$= \int_0^2 \int_1^3 x\, y^2 \left(\frac{4}{2} - \frac{1}{2}\right) dy\, dx$$

$$= \frac{3}{2}\int_0^2\int_1^3 xy^2\,dy\,dx$$

$$= \frac{3}{2}\int_0^2 x\left[\frac{y^3}{3}\right]_1^3 dx$$

$$= \frac{3}{2\times 3}\int_0^2 x\,[27-1]\,dx$$

$$= \frac{26}{2}\int_0^2 x\,dx = \frac{26}{2}\left[\frac{x^2}{2}\right]_0^2$$

$$= \frac{13}{2}(4-0)$$

$$= 26$$

2]. Evaluate $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz$

Soln.

$$\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz = \int_0^1\int_0^1\int_0^1 e^x e^y e^z\,dx\,dy\,dz$$

$$= \int_0^1 e^x\,dx \int_0^1 e^y\,dy \int_0^1 e^z\,dz$$

$$= (e^x)_0^1\,(e^y)_0^1\,(e^z)_0^1$$

$$= (1-0)\;(1-0)\;(1-0)$$

$$= 1$$

3]. Evaluate $\int_0^a\int_0^b\int_0^c xyz\,dx\,dy\,dz.$

Soln.

$$\int_0^a\int_0^b\int_0^c xyz\,dx\,dy\,dz = \int_0^a\int_0^b \left(\frac{x^2}{2}\right)_0^c yz\,dy\,dz$$

$$= \int_0^a\int_0^b \left(\frac{c^2}{2}-0\right) yz\,dy\,dz$$

$$= \frac{c^2}{2} \int_0^a \left(\frac{y^2}{2}\right)_0^b z \, dz$$

$$= \frac{c^2}{2} \int_0^a \left(\frac{b^2}{2} - 0\right) z \, dz$$

$$= \frac{b^2c^2}{4} \int_0^a z \, dz = \frac{b^2c^2}{4} \left[\frac{z^2}{2}\right]_0^a$$

$$= \frac{a^2b^2c^2}{8}$$

4]. Evaluate $\int_0^1 \int_0^y \int_0^{x+y} dx\, dy\, dz$

Soln.

$$\int_0^1 \int_0^y \int_0^{x+y} dz\, dx\, dy = \int_0^1 \int_0^y [z]_0^{x+y} dx\, dy$$

(correct form

$$= \int_0^1 \int_0^y [x+y-0]\, dx\, dy$$

$$= \int_0^1 \left[\frac{x^2}{2} + yx\right]_{x=0}^y dy$$

$$= \int_0^1 \left[\left(\frac{y^2}{2} + y^2\right) - 0\right] dy$$

$$= \frac{3}{2} \int_0^1 y^2\, dy$$

$$= \frac{3}{2}\left[\frac{y^3}{3}\right]_0^1 = \frac{1}{2}[1-0]$$

$$= \frac{1}{2}$$