3]. Find the volume of the tetrahedron bound by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Soln.

Given $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

$$\frac{x}{a}=1-\frac{y}{b}-\frac{z}{c}$$

$$x=a\left(1-\frac{y}{b}-\frac{z}{c}\right)$$

Take $x=0$,

$$\frac{y}{b}+\frac{z}{c}=1$$

$$\frac{y}{b}=1-\frac{z}{c}$$

$$y=b\left(1-\frac{z}{c}\right)$$

Take $x=0$ & $y=0$

$$\frac{z}{c}=1$$

$$z=c$$

Limits :

x : 0 to $a\left(1-\frac{y}{b}-\frac{z}{c}\right)$

y : 0 to $b\left(1-\frac{z}{c}\right)$

z : 0 to c

$$\text{volume} = \iiint_V dx\,dy\,dz$$

$$= \int_0^c \int_0^{b\left(1-\frac{z}{c}\right)} \int_0^{a\left(1-\frac{y}{b}-\frac{z}{c}\right)} dx\,dy\,dz$$

$$= \int_0^c \int_0^{b\left(1-\frac{z}{c}\right)} [x]_0^{a\left(1-\frac{y}{b}-\frac{z}{c}\right)} dy\,dz$$

$$= \int_0^c \int_0^{b\left(1-\frac{z}{c}\right)} \left[a\left(1-\frac{y}{b}-\frac{z}{c}\right)-0\right] dy\,dz$$

$$= a\int_0^c \int_0^{b\left(1-\frac{z}{c}\right)} \left[\left(1-\frac{z}{c}\right)-\frac{y}{b}\right] dy\,dz$$

$$= a\int_0^c \left[\left(1-\frac{z}{c}\right)y-\frac{y^2}{2b}\right]^{b\left(1-\frac{z}{c}\right)} dz$$

$$= a\int_0^c \left[\left(1-\frac{z}{c}\right) b\left(1-\frac{z}{c}\right) - \frac{1}{2b} b^2\left(1-\frac{z}{c}\right)^2\right] dz$$

$$= a\int_0^c \left[\left(1-\frac{z}{c}\right)^2 b - \frac{b}{2}\left(1-\frac{z}{c}\right)^2\right] dz$$

$$= \frac{ab}{2}\int_0^c \left(1-\frac{z}{c}\right)^2 dz$$

$$= \frac{ab}{2}\left[\frac{\left(1-\frac{z}{c}\right)^3}{\left(\frac{-1}{c}\right)3}\right]_0^c$$

$$= \frac{-abc}{6}\left[0-1\right]$$

$$\text{volume} = \frac{abc}{6}$$