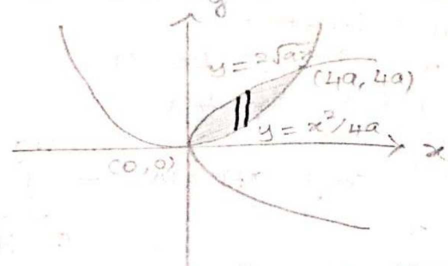
4]. Evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\,dy\,dx$ using change of order of integration.

Soln.:

Given $\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\,dy\,dx$



Order: dy dx (vertical strip)

limits:

x : 0 to $4a$

y : $\frac{x^2}{4a}$ to $2\sqrt{ax}$

Given $y = x^2/4a$ & $y = 2\sqrt{ax}$

$\therefore x^2 = 4ay \qquad y^2 = 4ax$

Now $\frac{x^2}{4a} = 2\sqrt{ax}$

$\frac{x^4}{16a^2} = 4ax$

$x^4 = 64a^3x$

$x^4 - 64a^3x = 0$

$x(x^3 - 64a^3) = 0$

$x = 0, \quad x^3 = 64a^3$

$x = 4a$

when $x=0$, $y=0$.

$x = 4a$, $y = \frac{16a^2}{4a} = 4a$

Intersection points are $(0,0)$ & $(4a, 4a)$

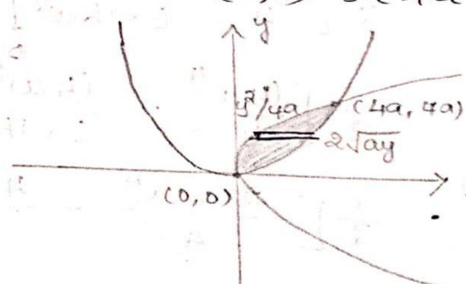
After changing the order of integration,

New order : dx dy (horizontal strip)

New limits:

x : $y^2/4a$ to $2\sqrt{ay}$

y : 0 to $4a$



$$\therefore \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\,dy\,dx = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} xy\,dx\,dy$$

$$= \int_0^{4a} \left[\frac{x^2}{2}\right]_{x=y^2/4a}^{2\sqrt{ay}} y\,dy$$

$$= \frac{1}{2}\int_0^{4a} \left[4ay - \frac{y^4}{16a^2}\right] y\, dy$$

$$= \frac{1}{2}\int_0^{4a} \left[4ay^2 - \frac{y^5}{16a^2}\right] dy$$

$$= \frac{1}{2}\left[4a\frac{y^3}{3} - \frac{y^6}{16\times 6\, a^2}\right]_0^{4a}$$

$$= \frac{1}{2}\left[\left(4a\frac{(4a)^3}{3} - \frac{(4a)^6}{96a^2}\right) - 0\right]$$

$$= \frac{1}{2}\left[\frac{4^4a^4}{3} - \frac{4^6a^6}{96a^2}\right]$$

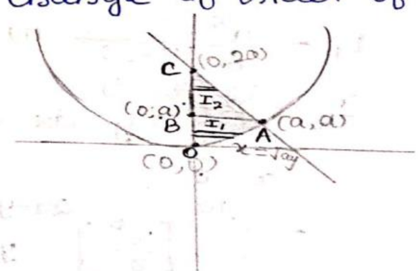
$$= \frac{4^4a^4}{2}\left[\frac{1}{3} - \frac{4^2}{96}\right] = \frac{256a^4}{2}\left[\frac{1}{3} - \frac{16}{96}\right]$$

$$= \frac{256a^4}{2}\left[\frac{1}{3} - \frac{1}{6}\right] = \frac{256a^4}{2}\left[\frac{2-1}{6}\right]$$

$$= \frac{256a^4}{12}$$

$$= \frac{64a^4}{3}$$

5] Evaluate $\int_0^a \int_{x^2/a}^{2a-x} xy\, dy\, dx$ using change of order of integration.



Soln.
Given $\int_0^a \int_{x^2/a}^{2a-x} xy\, dy\, dx$

Order: dy dx (vertical strip)

Limits:

x : 0 to a

y : $\frac{x^2}{a}$ to $2a-x$

Given $y = x^2/a$ & $y = 2a-x$

$x^2 = ay$ $\quad x+y = 2a$

when $x = 0$, $y = 2a$

$x = a$, $y = a$

$\therefore$ Intersection points $(0, 2a)$ & (a, a)

After changing the order of integration

New order : $dx\,dy$

The region of integration can be split up into two portions.

i.e., i). OAB and ii). ABC

$$I = I_1 + I_2$$

i). In the region OAB,

limits :

$x : 0$ to $\sqrt{ay}$

$y : 0$ to a

ii). In the region ABC

limits :

$x : 0$ to $2a-y$

$y : a$ to $2a$

$$I_1 = \int_0^a \int_0^{\sqrt{ay}} xy\,dx\,dy$$

$$= \int_0^a \left[\frac{x^2}{2}\right]_{x=0}^{\sqrt{ay}} y\,dy$$

$$= \int_0^a \left[\frac{ay}{2} - 0\right] y\,dy$$

$$= \int_0^a \frac{ay^2}{2}\,dy = \frac{a}{2}\left[\frac{y^3}{3}\right]_0^a$$

$$I_1 = \frac{a^4}{6} \longrightarrow (1)$$

$$I_2 = \int_a^{2a} \int_0^{2a-y} xy\,dx\,dy$$

$$= \int_a^{2a} \left[\frac{x^2}{2}\right]_{x=0}^{2a-y} y\,dy$$

$$= \frac{1}{2}\int_a^{2a} (2a-y)^2\,y\,dy = \frac{1}{2}\int_a^{2a} \left[4a^2+y^2-4ay\right] y\,dy$$

$$= \frac{1}{2}\int_{a}^{2a}\left[4a^2y + y^3 - 4ay^2\right] dy$$

$$= \frac{1}{2}\left[4a^2\frac{y^2}{2} + \frac{y^4}{4} - 4a\frac{y^3}{3}\right]_{a}^{2a}$$

$$= \frac{1}{2}\left[\left(\frac{4a^2(4a^2)}{2} + \frac{16a^4}{4} - \frac{4a}{3}8a^3\right) - \left(2a^4 + \frac{a^4}{4} - \frac{4a^4}{3}\right)\right]$$

$$= \frac{1}{2}\left[\left(8a^4 + 4a^4 - \frac{32}{3}a^4\right) - \left(\frac{9a^4}{4} - \frac{4a^4}{3}\right)\right]$$

$$= \frac{a^4}{2}\left[12 - \frac{32}{3} - \frac{9}{4} + \frac{4}{3}\right]$$

$$= \frac{a^4}{2}\left[\frac{144 - 128 - 27 + 16}{12}\right]$$

$$I_2 = \frac{5a^4}{24} \rightarrow (2)$$

From (1) & (2),

$$I = I_1 + I_2 = \frac{a^4}{6} + \frac{5a^4}{24}$$

$$= \frac{4a^4 + 5a^4}{24} = \frac{9a^4}{24}$$

$$\int_{0}^{a}\int_{x^2/a}^{2a-x} xy\, dy\, dx = \frac{3}{8}a^4$$

6]. Evaluate $\int_{0}^{1}\int_{x^2}^{2-x} xy\, dy\, dx$ using change of order of integration.

Soln.

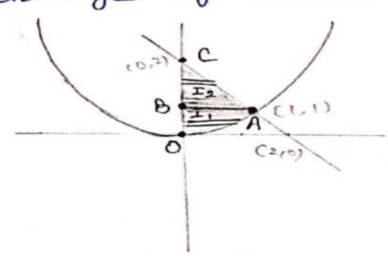
Given $\int_{0}^{1}\int_{x^2}^{2-x} xy\, dy\, dx$

Order: $dy\, dx$

Limits:

x : 0 to 1

y : x^2 to $2-x$



Given $y = x^2$ & $y = 2 - x$

$x^2 = y$ $\quad x + y = 2$

$\Rightarrow x^2 = 2 - x$

$x^2 + x - 2 = 0 \Rightarrow (x+2)(x-1) = 0$

$x = -2, 1$

when $x = 1$, $y = 1 \Rightarrow (1,1)$

After changing the order of integration,

New order : $dx\,dy$ (horizontal strip)

The region of integration can be split up into two portions.

i.e., i). OAB ii). ABC and $I = I_1 + I_2$

i). In the region OAB,

Limits :

$x : 0$ to $\sqrt{y}$

$y : 0$ to 1

$$I_1 = \int_0^1 \int_0^{\sqrt{y}} xy\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} y\,dy$$

$$= \int_0^1 \left(\frac{y}{2} - 0\right) y\,dy$$

$$= \int_0^1 \frac{y^2}{2}\,dy$$

$$= \left[\frac{y^3}{6}\right]_0^1$$

$$I_1 = \frac{1}{6} \longrightarrow (1)$$

ii). In the region ABC,

Limits :

$x : 0$ to $2-y$

$y : 1$ to 2

$$I_2 = \int_1^2 \int_0^{2-y} xy\,dx\,dy$$

$$= \int_1^2 \left[\frac{x^2}{2}\right]_0^{2-y} y\,dy$$

$$= \int_1^2 \left[\frac{(2-y)^2}{2} - 0\right] y\,dy$$

$$= \frac{1}{2}\int_1^2 \left[4+y^2-4y\right] y \, dy$$

$$= \frac{1}{2}\int_1^2 \left[4y+y^3-4y^2\right] dy$$

$$= \frac{1}{2}\left[\frac{4y^2}{2} + \frac{y^4}{4} - 4\frac{y^3}{3}\right]_1^2$$

$$= \frac{1}{2}\left[\left(2(4) + \frac{16}{4} - \frac{4}{3}(8)\right) - \left(2(1) + \frac{1}{4} - \frac{4}{3}\right)\right]$$

$$= \frac{1}{2}\left[8 + 4 - \frac{32}{3} - 2 - \frac{1}{4} + \frac{4}{3}\right]$$

$$= \frac{1}{2}\left[10 - \frac{32}{3} - \frac{1}{4} + \frac{4}{3}\right] = \frac{1}{2}\left[\frac{120-128-3+16}{12}\right]$$

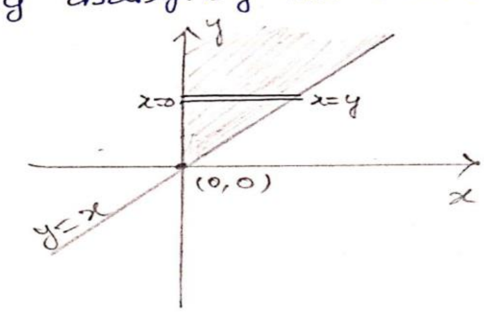
$$I_2 = \frac{5}{24} \longrightarrow (2)$$

From (1) and (2), $I = I_1 + I_2$

$$I = \frac{1}{6} + \frac{5}{24} = \frac{4+5}{24} = \frac{9}{24}$$

$$\therefore \int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx = \frac{3}{8}$$

7. Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dy \, dx$ by changing the order of integration.



Soln.

Given $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dy \, dx$

Order : dy dx (vertical strip)

Limits :

$x : 0$ to ∞

$y : x$ to ∞

After changing the order of integration,

New order : dx dy (Horizondal strip)

New limits : $x : 0$ to y

$y : 0$ to ∞

$$\therefore \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} \left[x\right]_0^y dy$$

$$= \int_0^\infty \frac{e^{-y}}{y}\, y\, dy$$

$$= \int_0^\infty e^{-y}\, dy$$

$$= \left[\frac{e^{-y}}{-1}\right]_0^\infty$$

$$= -\left[e^{-\infty} - 1\right] = -\left[0 - 1\right]$$

$$= 1$$