Change of order of Integration :

1. Evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dy\, dx$ using change of order of Integration.

Soln. :

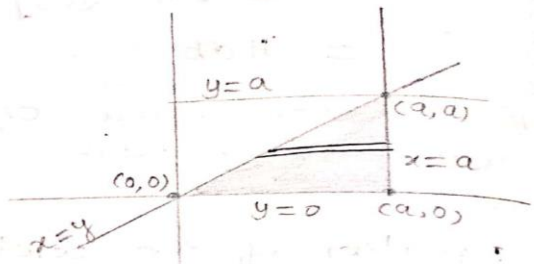
Given $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dy\, dx = \int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$ [correct form]

Order : $dx\, dy$

Limits :

x : y to a

y : 0 to a



$dx\,dy \to$ horizontal strip

x limit $\to$ interms of y

y limit $\to$ constant limits

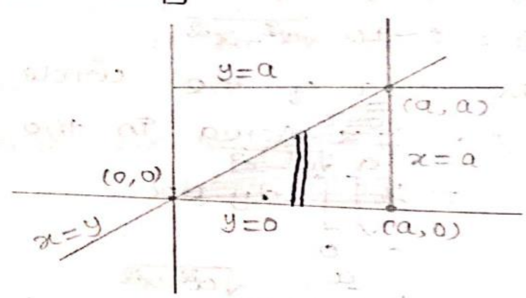
After changing the order of integration,

New order : $dy\, dx$ [Vertical strip]

New limits :

x : 0 to a

y : 0 to x



$$\therefore \int_0^a \int_y^a \left[\frac{x}{x^2+y^2}\right] dy\, dx$$

$$= \int_0^a \int_0^x \frac{dy}{y^2+x^2}\, x\, dx$$

$$= \int_0^a \left[\frac{1}{x}\tan^{-1}\left[\frac{y}{x}\right]\right]_{y=0}^{x} x\, dx$$

$$= \int_0^a \left[\frac{1}{x}\tan^{-1}(1) - 0\right] x\, dx$$

$$= \frac{\pi}{4}\int_0^a dx$$

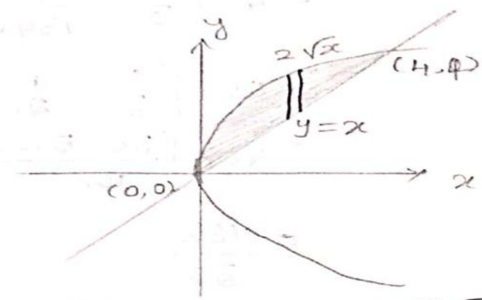
$$= \frac{\pi}{4}\left[x\right]_0^a$$

$$= \frac{\pi a}{4}$$

2]. Evaluate $\int_0^4 \int_x^{2\sqrt{x}} (x^2+y^2)\, dy\, dx$ using change of order of Integration.

Soln.:

Given $\int_0^4 \int_x^{2\sqrt{x}} (x^2+y^2)\, dy\, dx$



Order : dydx (vertical strip)

limits :

$x : 0$ to 4

$y : x$ to $2\sqrt{x}$

Given $y = x$ & $y = 2\sqrt{x}$

$x = 2\sqrt{x}$ $y^2 = 4x$

$x^2 = 4x \Rightarrow x^2 - 4x = 0$

$x(x-4) = 0$

$x = 0, 4$

Intersection point $(0,0), (4,4)$

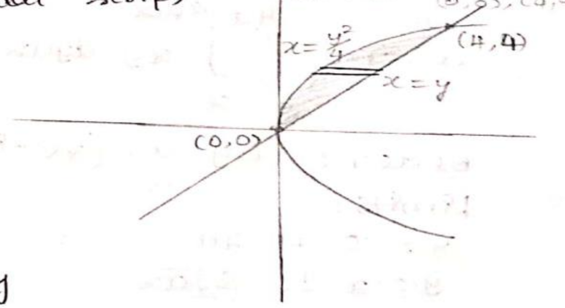
After changing the order of integration,

New order : dx dy (Horizontal strip)

New limits :

$x : y^2/4$ to y

$y : 0$ to 4



$$\therefore \int_0^4 \int_x^{2\sqrt{x}} (x^2+y^2)\, dy\, dx = \int_0^4 \int_{y^2/4}^{y} (x^2+y^2)\, dx\, dy$$

$$= \int_0^4 \left[\frac{x^3}{3} + y^2 x\right]_{x=y^2/4}^{y} dy$$

$$= \int_0^4 \left[\left(\frac{y^3}{3} + y^3\right) - \left(\frac{1}{3}\,\frac{y^6}{64} + \frac{y^4}{4}\right)\right] dy$$

$$= \int_0^4 \left[\frac{4}{3} y^3 - \frac{1}{192} y^6 - \frac{y^4}{4}\right] dy$$

$$= \left[\frac{4y^4}{3\times 4} - \frac{1}{192} \frac{y^7}{7} - \frac{y^5}{5\times 4}\right]_0^4$$

$$= \left(\frac{1}{3} 4^4 - \frac{1}{1344} 4^7 - \frac{1}{20} 4^5\right) - 0$$

$$= 4^4\left[\frac{1}{3} - \frac{4^3}{1344} - \frac{4}{20}\right] = 256\left[\frac{1}{3} - \frac{64}{1344} - \frac{1}{5}\right]$$

$$= 256\left[\frac{1}{3} - \frac{1}{21} - \frac{1}{5}\right]$$

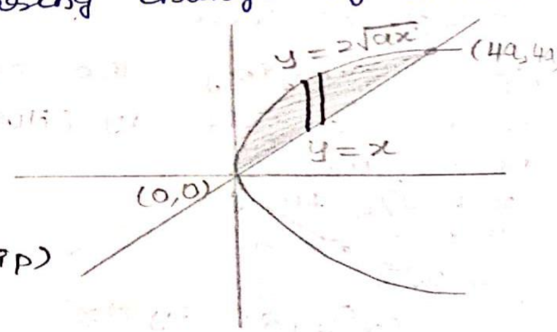
$$= 256\left[\frac{35-5-21}{105}\right] = \frac{2304}{105}$$

$$= \frac{768}{35}$$

3]. Evaluate $\int_0^{4a}\int_x^{2\sqrt{ax}} xy \, dy \, dx$ using change of order of integration.

Soln.:

Given $\int_0^{4a}\int_x^{2\sqrt{ax}} xy \, dy dx$



Order : dy dx (vertical strip)

Limits :

x : 0 to $4a$

y : x to $2\sqrt{ax}$

$y = x$ and $y = 2\sqrt{ax}$ $\Rightarrow$ $x = 2\sqrt{ax}$

$y^2 = 4ax$

$x^2 = 4ax$

$x^2 - 4ax = 0$

$x(x-4a) = 0$

$x = 0, \ x = 4a$

when $x = 0, \ y = 0$

$x = 4a, \ y = 4a$

Intersection points $(0,0)$ & $(4a, 4a)$

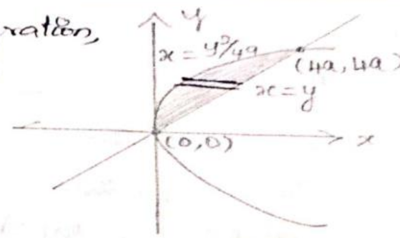
After changing the order of integration,

New order : dx dy (Horizontal strip)

New limits :

$x : y^2/4a$ to y

$y : 0$ to $4a$



$$\therefore \int_0^{4a}\int_x^{2\sqrt{ax}} xy\, dy\, dx = \int_0^{4a}\int_{y^2/4a}^{y} xy\, dx\, dy$$

$$= \int_0^{4a}\left[\int_{\frac{y^2}{4a}}^{y} x\, dx\right] y\, dy$$

$$= \int_0^{4a}\left[\frac{x^2}{2}\right]_{x = y^2/4a}^{y} y\, dy$$

$$= \frac{1}{2}\int_0^{4a}\left[y^2 - \left(\frac{y^2}{4a}\right)^2\right] y\, dy$$

$$= \frac{1}{2}\int_0^{4a}\left[y^2 - \frac{y^4}{16a^2}\right] y\, dy = \frac{1}{2}\int_0^{4a}\left[y^3 - \frac{y^5}{16a^2}\right] dy$$

$$= \frac{1}{2}\left[\frac{y^4}{4} - \frac{y^6}{6\times 16a^2}\right]_0^{4a}$$

$$= \frac{1}{2}\left[\frac{(4a)^4}{4} - \frac{(4a)^6}{6\times 16a^2}\right]$$

$$= \frac{1}{2}\left[\frac{\overset{64}{\cancel{256}}\, a^4}{\cancel{4}} - \frac{\overset{128}{\cancel{4096}}\, a^6}{\underset{3}{\cancel{96}}\, a^2}\right] = \frac{a^4}{2}\left[64 - \frac{128}{3}\right]$$

$$= \frac{a^4}{2}\left[\frac{192 - 128}{3}\right] = a^4\left[\frac{64}{6}\right]$$

$$= \frac{32a^4}{3}$$