Unit – I

Multiple Integrals

Applications of multiple integrals are to find areas and volume of various bodies just by taking a little part of them into consideration.

In probability theory it is used to evaluate probabilities of two dimensional continuous random variable.

Double Integration [Cartesian coordinate]

A double integral is computed by repeated single variable integration, integrate w.r. to one variable treating the other variable as constant.

ie., $I = \iint_R f(x, y)\, dx\, dy$ where R is the region.

Formulae :

1. $\int x^n\, dx = \frac{x^{n+1}}{n+1}$
2. $\int k\, dx = kx$
3. $\int \frac{1}{x}\, dx = \log x$
4. $\int e^x\, dx = e^x$
5. $\int \sin x\, dx = -\cos x$
6. $\int \cos x\, dx = \sin x$
7. $\int \tan x\, dx = \log(\sec x)$ (or) $-\log(\cos x)$
8. $\int \cot x\, dx = \log(\sin x) = \log(\csc x)$
9. $\int \csc x\, dx = \log[\csc x - \cot x]$
10. $\int \sec x\, dx = \log[\sec x + \tan x]$

11]. $\int \sec^2 x \, dx = \tan x$

12]. $\int \csc^2 x \, dx = -\cot x$

13]. $\int \sec x \tan x \, dx = \sec x$

14]. $\int \csc x \cot x = -\csc x$

15]. $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1}\left[\frac{x}{a}\right]$

16]. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log\left[\frac{x-a}{x+a}\right]$

17]. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left[\frac{x}{a}\right]$

18]. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$

19]. $\int \frac{-1}{\sqrt{1-x^2}} dx = \cos^{-1} x$

20]. $\int \frac{1}{1+x^2} dx = \tan^{-1} x$

21]. $\int \frac{-1}{1+x^2} dx = \cot^{-1} x$

22]. $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x$

23]. $\int \frac{-1}{x\sqrt{x^2-1}} dx = \csc^{-1} x$

24]. $\int \sqrt{a^2-x^2} \, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$

25]. $\int \frac{dx}{\sqrt{a^2+x^2}} = \log\left[x+\sqrt{x^2+a^2}\right]$

Problems on double integration in cartesian co-ordinates:

1]. Evaluate $\int_0^1 \int_1^2 x(x+y)\, dy\, dx$.

Soln.

Now $\int_0^1 \int_1^2 x(x+y)\, dy\, dx = \int_0^1 \left[\int_1^2 (x^2 + xy)\, dy \right] dx$

$= \int_0^1 \left[x^2 y + x\frac{y^2}{2} \right]_{y=1}^{2} dx$

$= \int_0^1 \left[\left(2x^2 + \frac{4x}{2}\right) - \left(x^2 + \frac{x}{2}\right) \right] dx$

$= \int_0^1 \left[2x^2 + 2x - x^2 - \frac{x}{2} \right] dx$

$= \int_0^1 \left[x^2 + \frac{3}{2} x \right] dx$

$= \left[\frac{x^3}{3} + \frac{3}{2}\frac{x^2}{2} \right]_{x=0}^{1}$

$= \left[\left(\frac{1}{3} + \frac{3}{4}\right) - 0 \right]$

$\int_0^1 \int_1^2 x(x+y)\, dy\, dx = \frac{13}{12}$.

2]. Evaluate $\int_0^a \int_0^b xy\,(x-y)\, dx\, dy$

Soln.

$\int_0^a \int_0^b xy\,(x-y)\, dx\, dy = \int_0^a \int_0^b \left[x^2 y - xy^2\right] dx\, dy$

$$= \int_0^a \left[\frac{x^3 y}{3} - \frac{x^2 y^2}{2} \right]_{x=0}^{b} dy$$

$$= \int_0^a \left[\frac{b^3}{3} y - \frac{b^2}{2} y^2 \right] dy$$

$$= \left[\frac{b^3}{3}\left(\frac{y^2}{2}\right) - \frac{b^2}{2}\left(\frac{y^3}{3}\right) \right]_{y=0}^{a}$$

$$= \left(\frac{b^3 a^2}{6} - \frac{b^2 a^3}{6} \right) - 0$$

$$\int_0^a \int_0^b xy(x-y)\,dx\,dy = \frac{a^2 b^2}{6}(b-a)$$

3]. Evaluate $\int_2^3 \int_1^2 \frac{1}{xy}\,dx\,dy$

Soln.

Now $\int_2^3 \int_1^2 \frac{1}{xy}\,dx\,dy = \int_2^3 \int_1^2 \frac{1}{x}\frac{1}{y}\,dx\,dy$

$$= \int_2^3 \frac{1}{y} [\log x]_{x=1}^{2}\,dy$$

$$= \int_2^3 \frac{1}{y} [\log 2 - \log 1]\,dy$$

$$= \int_2^3 \frac{1}{y} [\log 2]\,dy \quad [\because \log 1 = 0]$$

$$= \log 2 \int_2^3 \frac{dy}{y}$$

$$= \log 2 \, [\log y]_2^3$$

$$= \log 2 \, [\log 3 - \log 2]$$

$$\int_2^3 \int_1^2 \frac{1}{xy}\,dx\,dy = \log 2 \, \log\left[\frac{3}{2}\right]$$

4]. Evaluate $\int_0^3 \int_0^2 e^{x+y}\, dy\, dx$.

Soln. :

$$\int_0^3 \int_0^2 e^{x+y}\, dy\, dx = \int_0^3 \int_0^2 e^x e^y \, dy\, dx$$

$$= \int_0^3 e^x \left[e^y \right]_{y=0}^{2} dx$$

$$= \int_0^3 e^x \left[e^2 - e^0 \right] dx$$

$$= [e^2 - 1] \left[e^x \right]_{x=0}^{3}$$

$$= [e^2 - 1] \left[e^3 - e^0 \right]$$

$$= \underline{[e^2 - 1] \left[e^3 - 1 \right]}$$

5]. Evalute $\int_0^5 \int_0^{x^2} x(x^2+y^2)\, dx\, dy$

Soln. :

$$\int_0^5 \int_0^{x^2} x(x^2+y^2)\, dx\, dy = \int_0^5 \int_0^{x^2} [x^3 + xy^2]\, dy\, dx$$ [correct form]

$$= \int_0^5 \left[x^3 y + x\frac{y^3}{3} \right]_{y=0}^{x^2} dx$$

$$= \int_0^5 \left[\left(x^5 + \frac{x^7}{3} \right) - (0+0) \right] dx$$

$$= \int_0^5 \left[x^5 + \frac{x^7}{3} \right] dx$$

$$= \left[\frac{x^6}{6} + \frac{x^8}{24} \right]_0^5 = \left(\frac{5^6}{6} + \frac{5^8}{24} \right) - 0$$

$$= \underline{5^6 \left[\frac{29}{24} \right]}$$

6]. Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} dy\, dx$

Soln. :

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} dy\, dx = \int_0^a [y]_{y=0}^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \sqrt{a^2-x^2}\, dx$$

$$= \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \left(0 + \frac{a^2}{2}\sin^{-1}(1)\right) - (0+0)$$

$$= \frac{a^2}{2}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi a^2}{4}$$

7]. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2}$

Soln. :

$$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2} = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy}{1+x^2+y^2}\, dx \quad [\text{correct form}]$$

$$= \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy}{y^2+\left(\sqrt{1+x^2}\right)^2}\, dx$$

$$= \int_0^1 \left(\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left[\frac{y}{\sqrt{1+x^2}}\right]\right)_{y=0}^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}(1) - 0\right] dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \frac{\pi}{4} \, dx$$

$$= \frac{\pi}{4} \int_0^1 \frac{dx}{\sqrt{1+x^2}}$$

$$= \frac{\pi}{4} \left[\log \left(x + \sqrt{x^2+1^2} \right) \right]_0^1$$

$$= \frac{\pi}{4} \left[\log (1+\sqrt{2}) - \log (0+1) \right]$$

$$= \frac{\pi}{4} \log (1+\sqrt{2})$$

8]. Evaluate $\int_0^a \int_0^{\sqrt{ay}} xy \, dx \, dy$

Soln. :

$$\int_0^a \int_0^{\sqrt{ay}} xy \, dx \, dy = \int_0^a \left[\frac{x^2}{2} y \right]_{x=0}^{\sqrt{ay}} dy$$

$$= \int_0^a \left[\frac{ay}{2} y - 0 \right] dy$$

$$= \int_0^a \frac{ay^2}{2} \, dy$$

$$= \frac{a}{2} \left[\frac{y^3}{3} \right]_0^a$$

$$= \frac{a^4}{6}$$