5]. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dx\, dy\, dz$

Soln.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dx\, dy\, dz = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left[\frac{1-x^2-y^2}{2} - 0\right] dy\, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} [xy - x^3y - xy^3]\, dy\, dx$$

$$= \frac{1}{2} \int_0^1 \left[\frac{xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right]_{y=0}^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \int_0^1 \left[\left(\frac{x(1-x^2)}{2} - \frac{x^3(1-x^2)}{2} - \frac{x(1-x^2)^2}{4}\right) - 0\right] dx$$

$$= \frac{1}{2} \int_0^1 \left[\frac{[x - x^3 - x^3 + x^5]}{2} - \frac{x\cdot(1+x^4-2x^2)}{4}\right] dx$$

$$= \frac{1}{2(4)} \int_0^1 \left\{2[x - 2x^3 + x^5] - x - x^5 + 2x^3\right\} dx$$

$$= \frac{1}{8} \int_0^1 [2x - 4x^3 + 2x^5 - x - x^5 + 2x^3]\, dx$$

$$= \frac{1}{8} \int_0^1 [x - 2x^3 + x^5]\, dx$$

$$= \frac{1}{8} \left[\frac{x^2}{2} - \frac{2x^4}{4} + \frac{x^6}{6}\right]_0^1 = \frac{1}{8}\left[\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) - 0\right]$$

$$= \frac{1}{48}$$