Problems based on Area as a double integral in cartesian coordinates:

1. Find the area between enclosed by the curves $y^2 = 4x$ and $x^2 = 4y$

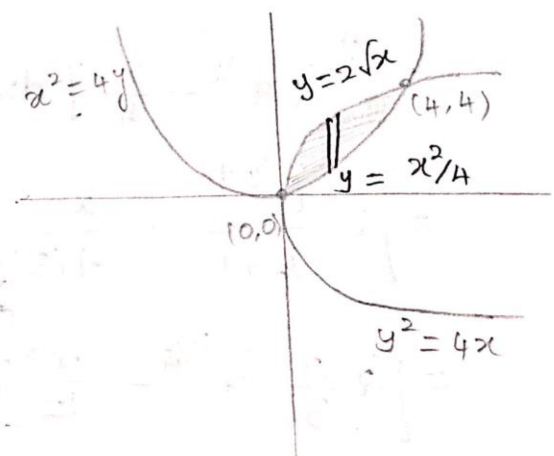
Soln.

Given that

$$y^2 = 4x \rightarrow (1)$$

$$x^2 = 4y \rightarrow (2)$$

$$y = \frac{x^2}{4}$$



Subs. y in (1),

$$\left(\frac{x^2}{4}\right)^2 = 4x$$

$$\frac{x^4}{16} = 4x$$

$$x^3 = 64$$

$$\boxed{x = 4}$$

dy dx → vertical strip
y limit → interms of x
x limit → constant limits

Subs. $x = 4$ in (2),

$$y = \frac{x^2}{4} = \frac{16}{4}$$

$$\boxed{y = 4}$$

Intersection point is $(4, 4)$

Limits:

$x : 0$ to 4

$y : \frac{x^2}{4}$ to $2\sqrt{x}$

we know that

$$A = \iint_R dy\,dx$$

$$= \int_0^4 \int_{x^2/4}^{2\sqrt{x}} dy\,dx$$

$$= \int_0^4 [y]_{y=x^2/4}^{2\sqrt{x}} dx = \int_0^4 \left[2\sqrt{x} - \frac{x^2}{4}\right] dx$$

$$= \int_0^4 \left[2x^{1/2} - \frac{x^2}{4}\right] dx$$

$$= \left[2\,\frac{x^{1/2+1}}{1/2+1} - \frac{x^3}{4(3)}\right]_0^4$$

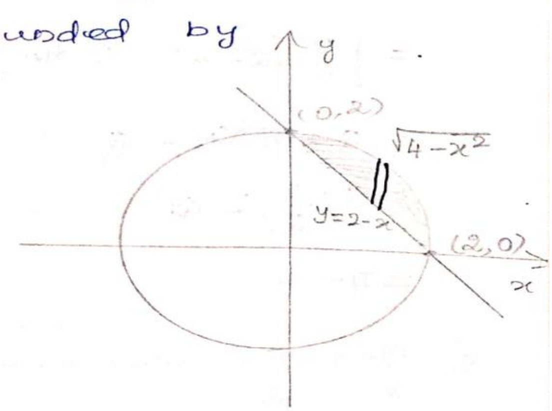
$$= \left[2\,\frac{2}{3}\,x^{3/2} - \frac{x^3}{12}\right]_0^4$$

$$= \left[\frac{4}{3}x^{3/2} - \frac{x^3}{12}\right]$$

$$= \left(\frac{4}{3}\,4^{3/2} - \frac{4^3}{12}\right) - 0 = \frac{4}{3}\,2^{2 \times 3/2} - \frac{64}{12}$$

$$= \frac{4}{3}(8) - \frac{16}{3} = \frac{32}{3} - \frac{16}{3} = \frac{16}{3}$$

2]. Find the smallest area bounded by $y = 2 - x$, $x^2 + y^2 = 4$



Soln.

Given that

$y = 2 - x \rightarrow (1)$

$x^2 + y^2 = 4 \rightarrow (2)$

subs. (1) in (2),

$x^2 + (2-x)^2 = 4$

$x^2 + 4 + x^2 - 4x = 4$

$2x^2 - 4x = 0$

$2x(x-2) = 0$

$\therefore x = 0, \; x = 2$

when $x = 0, \; y = 2 - 0 = 2$

$x = 2, \; y = 2 - 2 = 0$

$\therefore$ Intersection points are $(0, 2), (2, 0)$

$dy\,dx \rightarrow$ vertical strip

y limit $\rightarrow$ in terms of x

x limit $\rightarrow$ constant limits

$\rightarrow x^2 + y^2 = 4$

$y^2 = 4 - x^2$

$\therefore y = \pm\sqrt{4-x^2}$

Limits :

$x : 0$ to 2

$y : 2 - x$ to $\sqrt{4-x^2}$

$$\text{Area} = \iint_R dy\,dx$$

$$= \int_0^2 \int_{2-x}^{\sqrt{4-x^2}} dy\,dx$$

$$= \int_0^2 [y]_{2-x}^{\sqrt{4-x^2}} dx$$

$$= \int_0^2 \left[\sqrt{4-x^2} - (2-x)\right] dx$$

$$= \int_0^2 \sqrt{2^2-x^2}\,dx - \int_0^2 (2-x)\,dx$$

$$= \left[\frac{x}{2}\sqrt{2^2-x^2} + \frac{2^2}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2 - \left[2x - \frac{x^2}{2}\right]_0^2$$

$$= \left[\frac{2}{2}\sqrt{2^2 - 2^2} + 2\sin^{-1}\left(\frac{2}{2}\right)\right] - 0 - \left[\left(4 - \frac{4}{2}\right) - 0\right]$$

$$= 2\sin^{-1}(1) - 2$$

$$= 2\left(\frac{\pi}{2}\right) - 2$$

$$= \pi - 2$$

3]. Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, using double integration.

Soln.: Given that

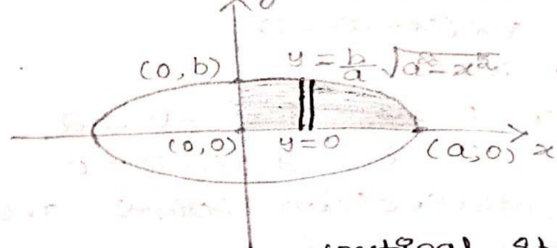
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = b^2\left[1 - \frac{x^2}{a^2}\right]$$

$$y^2 = \frac{b^2}{a^2}\left[a^2 - x^2\right]$$

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$



dy dx → vertical strip
y limit → in terms of x
x limit → constant limits

Limits:

$x : 0$ to a

$y : 0$ to $\frac{b}{a}\sqrt{a^2 - x^2}$

The area of the ellipse is,

$A = 4 \times$ Area in the first quadrant

$$= 4\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2 - x^2}} dy\, dx$$

$$= 4\int_0^a [y]_0^{\frac{b}{a}\sqrt{a^2 - x^2}} dx$$

$$= 4\int_0^a \left[\frac{b}{a}\sqrt{a^2 - x^2} - 0\right] dx$$

$$= \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\, dx$$

$$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\, dx$$

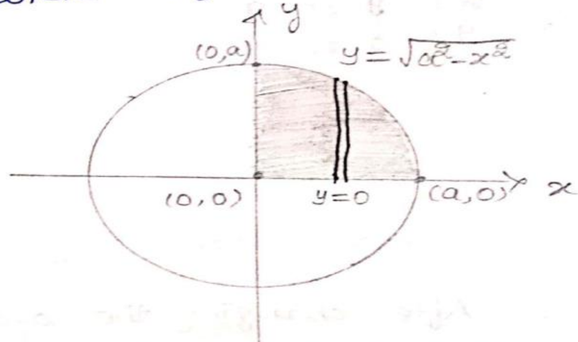
$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{4b}{a}\left[\left(0 + \frac{a^2}{2}\sin^{-1}(1)\right) - 0\right]$$

$$= \frac{4b}{a}\left[\frac{a^2}{2}\left(\frac{\pi}{2}\right)\right]$$

$$= \pi ab$$

4]. Find the area of a circle of radius by double integration.



Soln.:

Equation of the circle is

$$x^2+y^2=a^2$$

$$y^2=a^2-x^2$$

$$y=\pm\sqrt{a^2-x^2}$$

Limits:

x : 0 to a

y : 0 to $\sqrt{a^2-x^2}$

dy dx → vertical strip
y limit → interms of x
x limit → constant limits

The area of the circle is,

$A = 4 \times$ Area in the 1st quadrant

$$= 4\int_0^a\int_0^{\sqrt{a^2-x^2}} dy\, dx$$

$$= 4\int_0^a [y]_0^{\sqrt{a^2-x^2}}\, dx$$

$$= 4\int_0^a \left[\sqrt{a^2-x^2} - 0\right] dx$$

$$= 4\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a = 4\left[\left[0+\frac{a^2}{2}\sin^{-1}(1)\right]-0\right]$$

$$= 2a^2\,\frac{\pi}{2}$$

$$A = \pi a^2$$